The Annals of Probability
2010, Vol. 38, No. 2, 498–531
DOI: 10.1214/09-AOP494

A STOCHASTIC DIFFERENTIAL GAME FOR THE INHOMOGENEOUS ∞-LAPLACE EQUATION

BY RAMI ATAR AND AMARJIT BUDHIRAJA[1]

Technion—Israel Institute of Technology and University of North Carolina

Given a bounded $\mathcal{C}^2$ domain $G \subset \mathbb{R}^m$, functions $g \in \mathcal{C}(\partial G, \mathbb{R})$ and $h \in \mathcal{C}(\overline{G}, \mathbb{R} \setminus \{0\})$, let u denote the unique viscosity solution to the equation $-2\Delta_\infty u = h$ in G with boundary data g. We provide a representation for u as the value of a two-player zero-sum stochastic differential game.

1. Introduction.

1.1. *Infinity-Laplacian and games.* For an integer $m \geq 2$, let a bounded $\mathcal{C}^2$ domain $G \subset \mathbb{R}^m$, functions $g \in \mathcal{C}(\partial G, \mathbb{R})$ and $h \in \mathcal{C}(\overline{G}, \mathbb{R} \setminus \{0\})$ be given. We study a two-player zero-sum stochastic differential game (SDG), defined in terms of an m-dimensional state process that is driven by a one-dimensional Brownian motion, played until the state exits the domain. The functions g and h serve as terminal, and, respectively, running payoffs. The players' controls enter in a diffusion coefficient and in an unbounded drift coefficient of the state process. The dynamics are degenerate in that it is possible for the players to completely switch off the Brownian motion. We show that the game has value, and characterize the value function as the unique viscosity solution u (uniqueness of solutions is known from [10]) of the equation

$$\begin{cases} -2\Delta_\infty u = h, & \text{in } G, \\ u = g, & \text{on } \partial G. \end{cases} \tag{1.1}$$

Here, Δ_∞ is the infinity-Laplacian defined as $\Delta_\infty f = (Df)'(D^2 f)(Df)/|Df|^2$, provided $Df \neq 0$, where for a $\mathcal{C}^2$ function f we denote by Df the gradient and by $D^2 f$ the Hessian matrix. Our work is motivated by a representation for u of Peres et al. [10] (established in fact in a far greater generality), as the limit, as $\varepsilon \to 0$, of the value function V^ε of a discrete time random turn game, referred to as *Tug-of-War*, in which ε is a parameter. The contribution of the current work is the identification of a game for which the value function is precisely equal to u.

The infinity-Laplacian was first considered by Aronsson [1] in the study of absolutely minimal (AM) extensions of Lipschitz functions. Given a Lipschitz

Received August 2008; revised January 2009.

[1]Supported in part by Army Research Office Grant W911NF-0-1-0080.

AMS 2000 subject classifications. 91A15, 91A23, 35J70, 49L20.

Key words and phrases. Stochastic differential games, infinity-Laplacian, Bellman–Isaacs equation.

function u defined on the boundary ∂G of a domain G, a Lipschitz function $\widehat{u}$ extending u to $\overline{G}$ is called an AM extension of u if, for every open $U \subset G$, $\mathrm{Lip}_{\overline{U}}\, \widehat{u} = \mathrm{Lip}_{\partial U}\, u$, where for a real function f defined on $F \subset \mathbb{R}^m$, $\mathrm{Lip}_F\, f = \sup_{x,y \in F, x \neq y} |f(x) - f(y)|/|x-y|$. It was shown in [1] that a Lipschitz function $\widehat{u}$ on $\overline{G}$ that is $\mathcal{C}^2$ on G is an AM extension of $\widehat{u}|_{\partial G}$ if and only if $\widehat{u}$ is infinity-harmonic, namely satisfies $\Delta_\infty \widehat{u} = 0$ in G. This connection enables in some cases to prove uniqueness of AM extensions via PDE tools. However, due to the degeneracy of this elliptic equation, classical PDE approach in general is not applicable. Jensen [8] showed that an appropriate framework is through the theory of viscosity solutions, by establishing existence and uniqueness of viscosity solutions to the homogeneous version ($h = 0$) of (1.1), and showing that if g is Lipschitz then the solution is an AM extension of g. In addition to the relation to AM extensions, the infinity-Laplacian arises in a variety of other situations [4]. Some examples include models for sand-pile evolution [2], motion by mean curvature and stochastic target problems [9, 11].

We do not treat the homogenous equation for reasons mentioned later in this section. The inhomogeneous equation may admit multiple solutions when h assumes both signs [10]. Our assumption on h implies that either $h > 0$ or $h < 0$. Uniqueness for the case where these strict inequalities are replaced with weak inequalities is unknown [10]. Thus, the assumption we make on h is the minimal one under which uniqueness is known to hold in general (except the case $h = 0$).

Let us describe the Tug-of-War game introduced in [10]. Fix $\varepsilon > 0$. Let a token be placed at $x \in G$, and set $X_0 = x$. At the kth step of the game ($k \geq 1$), an independent toss of a fair coin determines which player takes the turn. The selected player is allowed to move the token from its current position $X_{k-1} \in G$ to a new position X_k in $\overline{G}$, in such a way that $|X_k - X_{k-1}| \leq \varepsilon$ ([10] requires $|X_k - X_{k-1}| < \varepsilon$ but this is an equivalent formulation in the setting described here). The game ends at the first time K when $X_K \in \partial G$. The associated payoff is given by

$$\mathbf{E}\left[g(X_K) + \frac{\varepsilon^2}{4} \sum_{k=0}^{K-1} h(X_k) \right]. \tag{1.2}$$

Player I attempts to maximize the payoff and player II's goal is to minimize it. It is shown in [10] that the value of the game, defined in a standard way and denoted $V^\varepsilon(x)$, exists, that V^ε converges uniformly to a function V referred to as the "continuum value function" and that V is the unique viscosity solution of (1.1) (these results are in fact also proved for the homogeneous case, and in generality greater than the scope of the current paper). The question of associating a game directly with the continuum value was posed and some basic technical challenges associated with it were discussed in [10].

Our approach to the question above is via a SDG formulation. To motivate the form of the SDG, we start with the Tug-of-War game and present some formal calculations (a precise definition of the SDG will appear later). Let $\{\xi_k, k \in \mathbb{N}\}$ be a sequence of i.i.d. random variables on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with $\mathbf{P}(\xi_k = 1) = \mathbf{P}(\xi_k = -1) = 1/2$, interpreted as the sequence of coin tosses. Let $\{\mathcal{F}_k\}_{k\geq 0}$ be a filtration of $\mathcal{F}$ to which $\{\xi_k\}$ is adapted and such that $\{\xi_{k+1}, \xi_{k+2}, \dots\}$ is independent of $\mathcal{F}_k$ for every $k \geq 0$. Let $\{a_k\}$, $\{b_k\}$ be $\{\mathcal{F}_k\}$-predictable sequences of random variables with values in $\overline{\mathbb{B}_\varepsilon(0)} = \{x \in \mathbb{R}^m : |x| \leq \varepsilon\}$. These sequences correspond to control actions of players I and II; that is, a_k (resp., b_k) is the displacement exercised by player I (resp., player II) if it wins the kth coin toss. Associating the event $\{\xi_k = 1\}$ with player I winning the kth toss, one can write the following representation for the position of the token, starting from initial state x. For $j \in \mathbb{N}$,

$$X_j = x + \sum_{k=1}^{j}\left[a_k \frac{1+\xi_k}{2} + b_k \frac{1-\xi_k}{2}\right] = \sum_{k=1}^{j} \frac{a_k - b_k}{2}\xi_k + \sum_{k=1}^{j} \frac{a_k + b_k}{2}.$$

We shall refer to $\{X_j\}$ as the "state process." This representation, in which turns are not taken at random but both players select an action at each step, and the noise enters in the dynamics, is more convenient for the development that follows. Let $\varepsilon = 1/\sqrt{n}$ and rescale the control processes by defining, for $t \geq 0$, $A_t^n = \sqrt{n} a_{[nt]}$, $B_t^n = \sqrt{n} b_{[nt]}$. Consider the continuous time state process $X_t^n = X_{[nt]}$, and define $\{W_t^n\}_{t\geq 0}$ by setting $W_0^n = 0$ and using the relation

$$W_t^n = W_{(k-1)/n}^n + \left(t - \frac{k-1}{n}\right)\sqrt{n}\xi_k, \qquad t \in \left(\frac{k-1}{n}, \frac{k}{n}\right], k \in \mathbb{N}.$$

Then we have

$$X_t^n = x + \frac{1}{2}\int_0^t (A_s^n - B_s^n)\, dW_s^n + \frac{1}{2}\int_0^t \sqrt{n}(A_s^n + B_s^n)\, ds. \tag{1.3}$$

Note that W^n converges weakly to a standard Brownian motion, and since $|A_t^n| \vee |B_t^n| \leq 1$, the second term on the right-hand side of (1.3) forms a tight sequence. Thus, it is easy to guess a substitute for it in the continuous game. Interpretation of the asymptotics of the third term is more subtle, and is a key element of the formulation. One possible approach is to replace the factor $\sqrt{n}$ by a large quantity that is dynamically controlled by the two players. This point of view motivates one to consider the identity (that we prove in Proposition 5.1)

$$\begin{aligned} -2\Delta_\infty f = \sup_{|b|=1, d\geq 0} \inf_{|a|=1, c\geq 0} \bigg\{ &-\frac{1}{2}(a-b)'(D^2 f)(a-b) \\ &- (c+d)(a+b)\cdot Df \bigg\}, \end{aligned} \tag{1.4}$$
$$f \in \mathcal{C}^2, Df \neq 0,$$

for the following reason. Let $\mathcal{H} = \mathcal{S}^{m-1} \times [0, \infty)$ where $\mathcal{S}^{m-1}$ is the unit sphere in $\mathbb{R}^m$. The expression in curly brackets is equal to $\mathcal{L}^{a,b,c,d} f(x)$, where for

$(a,c),(b,d)\in\mathcal{H}$, $\mathcal{L}^{a,b,c,d}$ is the controlled generator associated with the process

$$(1.5)\quad X_t = x + \int_0^t (A_s - B_s)\,dW_s + \int_0^t (C_s + D_s)(A_s + B_s)\,ds, \qquad t\in[0,\infty),$$

and (A,C) and (B,D) are control processes taking values in $\mathcal{H}$. Since Δ_∞ is related to (1.3) via the Tug-of-War, and $\mathcal{L}^{a,b,c,d}$ to (1.5), identity (1.4) suggests to regard (1.5) as a formal limit of (1.3). Consequently the SDG will have (1.5) as a state process, where the controls (A,C) and (B,D) are chosen by the two players. Finally, the payoff functional, as a formal limit of (1.2), and accounting for the extra factor of $1/2$ in (1.3), will be given by $\mathbf{E}[\int_0^\tau h(X_s)\,ds + g(X_\tau)]$, where $\tau = \inf\{t : X_t \notin G\}$ (with an appropriate convention regarding $\tau=\infty$).

A precise formulation of this game is given in Section 1.2, along with a statement of the main result. Section 1.3 discusses the technique and some open problems.

Throughout, we will denote by $\mathbb{S}(m)$ the space of symmetric $m\times m$ matrices, and by $I_m \in \mathbb{S}(m)$ the identity matrix. A function $\vartheta : [0,\infty)\to[0,\infty)$ will be said to be a *modulus* if it is continuous, nondecreasing, and satisfies $\vartheta(0)=0$.

1.2. *SDG formulation and main result*. Recall that G is a bounded $\mathcal{C}^2$ domain in $\mathbb{R}^m$, and that $g : \partial G \to \mathbb{R}$ and $h : \overline{G} \to \mathbb{R}\setminus\{0\}$ are given continuous functions. In particular we have that either $h>0$ or $h<0$. Since the two cases are similar, we will only consider $h>0$, and use the notation $\underline{h} := \inf_{\overline{G}} h > 0$. Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbf{P})$ be a complete filtered probability space with right-continuous filtration, supporting an $(m+1)$-dimensional $\{\mathcal{F}_t\}$-Brownian motion $\overline{W} = (W,\widetilde{W})$, where W and $\widetilde{W}$ are one- and m-dimensional Brownian motions, respectively. Let $\mathbf{E}$ denote expectation with respect to $\mathbf{P}$. Let X_t be a process taking values in $\mathbb{R}^m$, given by

$$(1.6)\quad X_t = x + \int_0^t (A_s - B_s)\,dW_s + \int_0^t (C_s + D_s)(A_s + B_s)\,ds, \qquad t\in[0,\infty),$$

where $x\in\overline{G}$, A_t and B_t take values in the unit sphere $\mathcal{S}^{m-1}\subset\mathbb{R}^m$, and C_t and D_t take values in $[0,\infty)$. Denote

$$(1.7)\qquad Y^0 = (A,C), \qquad Z^0 = (B,D).$$

The processes Y^0 and Z^0 take values in $\mathcal{H} = \mathcal{S}^{m-1}\times[0,\infty)$. These processes will correspond to control actions of the maximizing and minimizing player, respectively. We remark that, although $\widetilde{W}$ does not appear explicitly in the dynamics (1.6), the control processes Y^0, Z^0 will be required to be $\{\mathcal{F}_t\}$-adapted, and thus may depend on it. In Section 1.3, we comment on the need for including this auxiliary Brownian motion in our formulation. Let

$$\tau = \inf\{t : X_t \in \partial G\}.$$

Throughout, we will follow the convention that the infimum over an empty set is ∞. We write

$$(1.8)\qquad X(x,Y^0,Z^0) \qquad [\text{resp., } \tau(x,Y^0,Z^0)]$$

for the process X (resp., the random time τ) when it is important to specify the explicit dependence on (x, Y^0, Z^0). If $\tau < \infty$ a.s., then the payoff $J(x, Y^0, Z^0)$ is well defined with values in $(-\infty, \infty]$, where

$$J(x, Y^0, Z^0) = \mathbf{E}\left[\int_0^\tau h(X_s)\,ds + g(X_\tau)\right] \tag{1.9}$$

and X is given by (1.6). When $\mathbf{P}(\tau(x, Y^0, Z^0) = \infty) > 0$, we set $J(x, Y^0, Z^0) = \infty$, in agreement with the expectation of the first term in (1.9).

We turn to the precise definition of the SDG. For a process $H^0 = (A, C)$ taking values in $\mathcal{H}$, we let $S(H^0) = \operatorname{ess\,sup} \sup_{t\in[0,\infty)} C_t$. In the formulation below, each player initially declares a bound S, and then plays so as to keep $S(H^0) \le S$.

DEFINITION 1.1. (i) A pair $H = (\{H_t^0\}, S)$, where $S \in \mathbb{N}$ and $\{H_t^0\}$ is a process taking values in $\mathcal{H}$, is said to be an admissible control if $\{H_t^0\}$ is $\{\mathcal{F}_t\}$-progressively measurable, and $S(H^0) \le S$. The set of all admissible controls is denoted by M. For $H = (\{H_t^0\}, S) \in M$, denote $\mathbf{S}(H) = S$.

(ii) A mapping $\varrho : M \to M$ is said to be a strategy if, for every t,

$$\mathbf{P}(H_s^0 = \widetilde{H}_s^0 \text{ for a.e. } s \in [0, t]) = 1 \quad \text{and} \quad S = \widetilde{S}$$

implies

$$\mathbf{P}(I_s^0 = \widetilde{I}_s^0 \text{ for a.e. } s \in [0, t]) = 1 \quad \text{and} \quad T = \widetilde{T},$$

where $(I^0, T) = \varrho[(H^0, S)]$ and $(\widetilde{I}^0, \widetilde{T}) = \varrho[(\widetilde{H}^0, \widetilde{S})]$. The set of all strategies is denoted by $\widetilde{\Gamma}$. For $\varrho \in \widetilde{\Gamma}$, let $\mathbf{S}(\varrho) = \sup_{H\in M} \mathbf{S}(\varrho[H])$. Let

$$\Gamma = \{\varrho \in \widetilde{\Gamma} : \mathbf{S}(\varrho) < \infty\}.$$

We will use the symbols Y and α for generic control and strategy for the maximizing player, and Z and β for the minimizing player. If $Y = (Y^0, K), Z = (Z^0, L) \in M$, we sometimes write $J(x, Y, Z) = J(x, (Y^0, K), (Z^0, L))$ for $J(x, Y^0, Z^0)$. Similar conventions will be used for $X(x, Y, Z)$ and $\tau(x, Y, Z)$. Let

$$J^x(Y, \beta) = J(x, Y, \beta[Y]), \qquad x \in \overline{G}, Y \in M, \beta \in \Gamma,$$
$$J^x(\alpha, Z) = J(x, \alpha[Z], Z), \qquad x \in \overline{G}, \alpha \in \Gamma, Z \in M.$$

Define analogously $X^x(Y, \beta)$, $X^x(\alpha, Z)$, $\tau^x(Y, \beta)$ and $\tau^x(\alpha, Z)$ via (1.8). Define the lower value of the SDG by

$$V(x) = \inf_{\beta\in\Gamma} \sup_{Y\in M} J^x(Y, \beta) \tag{1.10}$$

and the upper value by

$$U(x) = \sup_{\alpha\in\Gamma} \inf_{Z\in M} J^x(\alpha, Z). \tag{1.11}$$

The game is said to have a value if $U = V$.

Recall that the infinity-Laplacian is defined by $\Delta_\infty f = p'\Sigma p/|p|^2$, where f is a $\mathcal{C}^2$ function, $p = Df$ and $\Sigma = D^2 f$, provided that $p \neq 0$. Thus, $\Delta_\infty f$ is equal to the second derivative in the direction of the gradient. In the special case where $D^2 f(x)$ is of the form λI_m for some real λ, it is therefore natural to define $\Delta_\infty f(x) = \lambda$ even if $Df(x) = 0$ [10]. This will be reflected in the definition of viscosity solutions of (1.1), that we state below. Let

$$\mathcal{D}_0 = \{(0, \lambda I_m) \in \mathbb{R}^m \times \mathcal{S}(m) : \lambda \in \mathbb{R}\},$$
$$\mathcal{D}_1 = (\mathbb{R}^m \setminus \{0\}) \times \mathcal{S}(m),$$
$$\mathcal{D} = \mathcal{D}_0 \cup \mathcal{D}_1$$

and

$$\Lambda(p, \Sigma) = \begin{cases} -2\lambda, & (p, \Sigma) = (0, \lambda I_m) \in \mathcal{D}_0, \\ -2\dfrac{p'\Sigma p}{|p|^2}, & (p, \Sigma) \in \mathcal{D}_1. \end{cases}$$

DEFINITION 1.2. A continuous function $u : \overline{G} \to \mathbb{R}$ is said to be a viscosity supersolution (resp., subsolution) of (1.1), if:

(i) for every $x \in G$ and $\varphi \in \mathcal{C}^2(G)$ for which $(p, \Sigma) := (D\varphi(x), D^2\varphi(x)) \in \mathcal{D}$, and $u - \varphi$ has a global minimum [maximum] on G at x, one has

$$\Lambda(p, \Sigma) - h(x) \geq 0 \qquad [\leq 0]; \tag{1.12}$$

and

(ii) $u = g$ on ∂G.

A viscosity solution is a function which is both a super- and a subsolution.

The result below has been established in [10].

THEOREM 1.1. *There exists a unique viscosity solution to* (1.1).

The following is our main result.

THEOREM 1.2. *The functions U and V are both viscosity solutions to* (1.1). *Consequently, the SDG has a value.*

In what follows, we use the terms subsolution, supersolution and solution as shorthand for viscosity subsolution, etc.

1.3. *Discussion*. We describe here our approach to proving the main result, and mention some obstacles in extending it.

A common approach to showing solvability of Bellman–Isaacs (BI) equations [(1.1) can be viewed as such an equation due to (1.4)] by the associated value function, is by proving that the value function satisfies a dynamic programming principle (DPP). Roughly speaking, this is an equation expressing the fact that, rather than attempting to maximize their profit by considering directly the payoff functional, the players may consider the payoff incurred up to a time t plus the value function evaluated at the position X_t that the state reaches at that time. Although in a single player setting (i.e., in pure control problems) DPP are well understood, game theoretic settings as in this paper are significantly harder. In particular, as we shall shortly point out, there are some basic open problems related to such DPP. In a setting with a finite time horizon, Fleming and Souganidis [7] established a DPP based on careful discretization and approximation arguments. We have been unable to carry out a similar proof in the current setting, which includes a payoff given in terms of an exit time, degenerate diffusion and unbounded controls.

Swiech [12] has developed an alternative approach to the above problem that relies on existence of solutions. Instead of establishing a DPP for the value function, the idea of [12] is to show that any *solution* must satisfy a DPP. To see what is meant by such a DPP and how it is used, consider the equation, $-2\Delta_\infty u + \lambda u = h$ in G, $u = g$ on ∂G, where $\lambda \geq 0$ is a constant, associated with the payoff in (1.9) modified by a discount factor. Assume that one can show that whenever u and v are sub- and supersolutions, respectively, then

$$u(x) \leq \sup_{\alpha\in\Gamma} \inf_{Z\in M} \mathbf{E}\biggl[\int_0^\sigma e^{-\lambda s} h(X_s)\,ds + e^{-\lambda\sigma} u(X_\sigma)\biggr], \tag{1.13}$$

$$v(x) \geq \sup_{\alpha\in\Gamma} \inf_{Z\in M} \mathbf{E}\biggl[\int_0^\sigma e^{-\lambda s} h(X_s)\,ds + e^{-\lambda\sigma} v(X_\sigma)\biggr], \tag{1.14}$$

for $X = X[x, \alpha[Z], Z]$, $\tau = \tau[x, \alpha[Z], Z]$ and $\sigma = \sigma(t) = \tau \wedge t$. Sending $t \to \infty$ in the above equations, one would formally obtain

$$u(x) \leq \sup_{\alpha\in\Gamma} \inf_{Z\in M} \mathbf{E}\biggl[\int_0^\tau e^{-\lambda s} h(X_s)\,ds + e^{-\lambda\tau} g(X_\tau)\biggr] \leq v(x), \tag{1.15}$$

in particular yielding that if $u = v$ is a solution to the equation then it must equal the upper value function. This would establish unique solvability of the equation by the upper value function, provided there exists a solution. In the case $\lambda > 0$, justifying the above formal limit is straightforward (see [12]) but the case $\lambda = 0$, as in our setting, requires a more careful argument. Our proofs exploit the uniform positivity of h due to which the minimizing player will not allow τ to be too large. This leads to uniform estimates on the decay of $\mathbf{P}(\tau > t)$ as $t \to \infty$, from which an inequality as in (1.15) follows readily. This discussion also explains why we are unable to treat the case $h = 0$.

Establishing DPP as in (1.13), (1.14) is thus a key ingredient in this approach. For a class of BI equations, defined on all of $\mathbb{R}^m$, for which the associated game has a bounded action set and a fixed, finite time horizon, such a DPP was proved in Swiech [12]. In the current paper, although we do not establish (1.13), (1.14) in the above form, we derive similar inequalities (for $\lambda = 0$) for a related bounded action game, defined on G. The characterization of the value function for the original unbounded action game is then treated by taking suitable limits.

Both [7] and [12] require some assumptions on the sample space and underlying filtration. In [7], the underlying filtration is the one generated by the driving Brownian motion. The approach taken in [12], which the current paper follows, allows for a general filtration as long as it is rich enough to support an m-dimensional Brownian motion, independent of the Brownian motion driving the state process [for example, it could be the filtration generated by an $(m+1)$-dimensional Brownian motion]. The reason for imposing this requirement in [12] is that inequalities similar to (1.13) and (1.14) are proved by first establishing them for a game associated with a nondegenerate elliptic equation, and then taking a vanishing viscosity limit. This technical issue is the reason for including the auxiliary process $\widetilde{W}$ in our formulation as well. As pointed out in [12], the question of validity of the DPP and the characterization of the value as the unique solution to the PDE, under an arbitrary filtration, remains a basic open problem on SDGs.

The unboundedness of the action space, on one hand, and the combination of degeneracy of the dynamics and an exit time criterion on the other hand, make it hard to adapt the results of [12] to our setting. In order to overcome the first difficulty, we approximate the original SDG by a sequence of games with bounded action spaces, that are more readily analyzed. For the bounded action game, existence of solutions to the upper and lower BI equations follow from [5]. We show that the solutions to these equations satisfy a DPP similar to (1.13) and (1.14) (Proposition 4.1). As discussed above, existence of solutions along with the DPP yields the characterization of these solutions as the corresponding value functions. Next, as we show in Lemma 2.5, the upper and lower value functions for the bounded action games approach the corresponding value functions of the original game, pointwise, as the bounds approach ∞. Moreover, in Lemma 2.4, we show that any uniform subsequential limit, as the bounds approach ∞, of solutions to the BI equation for bounded action games is a viscosity solution of (1.1). The last piece in the proof of the main result is then showing existence of uniform (subsequential) limits. This is established in Theorem 2.1 by proving equicontinuity, in the parameters governing the bounds, of the value functions for bounded action games. The proof of equicontinuity is the most technical part of this paper and the main place where the $\mathcal{C}^2$ assumption on the domain is used. This is also the place where the possibility of degenerate dynamics close to the exit time needs to be carefully analyzed.

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.2 based on results on BI equations for bounded action SDG. These results are established in Sections 3 (equicontinuity of the value functions) and 4 (relating the value function to the PDE). Finally, it is natural to ask whether the state process, obtained under δ-optimal play by both players, converges in law as δ tends to zero. Section 5 describes a recently obtained result [3] that addresses this issue.

2. Relation to Bellman–Isaacs equation. In this section, we prove Theorem 1.2 by relating the value functions U and V to value functions of SDG with bounded action sets, and similarly, the solution to (1.1) to that of the corresponding Bellman–Isaacs equations.

Let $p \in \mathbb{R}^m$, $p \neq 0$ and $S \in \mathbb{S}(m)$ be given, and, for $n \in \mathbb{N}$, fix $p_n \in \mathbb{R}^m$, $p_n \neq 0$ and $S_n \in \mathbb{S}(m)$, such that $p_n \to p$, $S_n \to S$. Denote $\overline{p} = p/|p|$ and $\overline{p}_n = p_n/|p_n|$. Let $\{k_n\}$ and $\{l_n\}$ be positive, increasing sequences such that $k_n \to \infty$, $l_n \to \infty$.

Denote

$$\Phi(a,b,c,d;p,S) = -\tfrac{1}{2}(a-b)'S(a-b) - (c+d)(a+b)\cdot p, \tag{2.1}$$

and let

$$\Lambda^+_{kl}(p,S) = \max_{|b|=1,0\le d\le l}\ \min_{|a|=1,0\le c\le k} \Phi(a,b,c,d;p,S), \tag{2.2}$$

$$\Lambda^-_{kl}(p,S) = \min_{|a|=1,0\le c\le k}\ \max_{|b|=1,0\le d\le l} \Phi(a,b,c,d;p,S). \tag{2.3}$$

Set

$$\Lambda^+_n(p,S) = \Lambda^+_{k_n l_n}(p,S), \qquad \Lambda^-_n(p,S) = \Lambda^-_{k_n l_n}(p,S).$$

LEMMA 2.1. *One has $\Lambda^+_n(p_n,S_n) \to \Lambda(p,S)$, and $\Lambda^-_n(p_n,S_n) \to \Lambda(p,S)$, as $n \to \infty$.*

PROOF. We prove only the statement regarding Λ^-_n, since the other statement can be proved analogously. We omit the superscript "$-$" from the notation. Denote $\Phi_n(a,b,c,d) = \Phi(a,b,c,d;p_n,S_n)$. Let

$$\overline{\Lambda}_n(a,c) = \max_{|b|=1,0\le d\le l_n} \Phi_n(a,b,c,d).$$

Let (a^*_n,c^*_n) be such that $\Lambda^*_n := \Lambda_n(p_n,S_n) = \overline{\Lambda}_n(a^*_n,c^*_n)$. Note that $\Lambda^*_n \le \overline{\Lambda}_n(\overline{p}_n,0)$, which is bounded from above as $n\to\infty$, since $(b+\overline{p}_n)\cdot\overline{p}_n \ge 0$ for all $b \in \mathcal{S}^{m-1}$, $n \ge 1$. On the other hand, if for some fixed $\varepsilon > 0$, $a^*_n \cdot p_n < |p_n| - \varepsilon$ holds for infinitely many n, then $\limsup \overline{\Lambda}_n(a^*_n,c_n) = \infty$ for any choice of c_n contradicting the statement that Λ^*_n is bounded from above. This shows, for every $\varepsilon > 0$,

$$|p_n| - \varepsilon \le a^*_n \cdot p_n \le |p_n|$$

for all large n. In particular, $a^*_n \to \overline{p}$. Next note that

$$\Lambda^*_n = \overline{\Lambda}_n(a^*_n,c^*_n) \ge \Phi_n(-\overline{p}_n,a^*_n,l_n,c^*_n) \ge -\tfrac{1}{2}(\overline{p}_n + a^*_n)'S_n(\overline{p}_n + a^*_n)$$

hence,

$$\liminf \Lambda_n^* \geq -2\overline{p}' S\overline{p} = \Lambda(p, S).$$

Also, with $(\widetilde{b}_n, \widetilde{d}_n) \in \arg\max_{(b,d)} \Phi_n(b, \overline{p}_n, d, k_n)$,

$$\begin{aligned} \Lambda_n^* &= \overline{\Lambda}_n(a_n^*, c_n^*) \leq \overline{\Lambda}_n(\overline{p}_n, k_n) \\ &= -\tfrac{1}{2}(\widetilde{b}_n - \overline{p}_n)' S_n(\widetilde{b}_n - \overline{p}_n) - (\widetilde{d}_n + k_n)(\widetilde{b}_n + \overline{p}_n) \cdot p_n \\ &\leq -\tfrac{1}{2}(\widetilde{b}_n - \overline{p}_n)' S_n(\widetilde{b}_n - \overline{p}_n). \end{aligned} \tag{2.4}$$

If $\widetilde{b}_n \to -\overline{p}$ does not hold, then $\liminf \Lambda_n^* = -\infty$ by the first line of (2.4) which contradicts the previous display. This shows $\widetilde{b}_n \to -\overline{p}$. Hence, from the second line of (2.4)

$$\limsup \Lambda_n^* \leq -2\overline{p}' S\overline{p} = \Lambda(p, S). \qquad \square$$

We now consider two formulations of SDG with bounded controls, the first being based on Definition 1.1 whereas the second is more standard. For $k, l \in \mathbb{N}$, let

$$\begin{aligned} M_k &= \{Y \in M : \mathbf{S}(Y) \leq k\}, \\ \Gamma_l &= \{\beta \in \Gamma : \mathbf{S}(\beta) \leq l\}. \end{aligned}$$

Define accordingly the lower value

$$V_{kl}(x) = \inf_{\beta \in \Gamma_l} \sup_{Y \in M_k} J^x(Y, \beta), \tag{2.5}$$

and the upper value

$$U_{kl}(x) = \sup_{\alpha \in \Gamma_k} \inf_{Z \in M_l} J^x(\alpha, Z). \tag{2.6}$$

DEFINITION 2.1. (i) A process $\{H_t\}$ taking values in $\mathcal{H}$ is said to be a simple admissible control if it is $\{\mathcal{F}_t\}$-progressively measurable. We denote by M^0 the set of all simple admissible controls, and let $M_k^0 = \{H \in M^0 : S(H) \leq k\}$.

(ii) Given $k, l \in \mathbb{N}$, we say that a mapping $\varrho : M_k^0 \to M_l^0$ is a simple strategy, and write $\varrho \in \Gamma_{kl}^0$ if, for every t,

$$\mathbf{P}(H_s = \widetilde{H}_s \text{ for a.e. } s \in [0, t]) = 1$$

implies

$$\mathbf{P}(\varrho[H]_s = \varrho[\widetilde{H}]_s \text{ for a.e. } s \in [0, t]) = 1.$$

For $\beta \in \Gamma^0_{kl}$, $Y \in M^0_k$, we write $J^x(Y, \beta(Y))$ as $J^x(Y, \beta)$. For $\alpha \in \Gamma^0_{lk}$, $Z \in M^0_l$, $J^x(\alpha, Z)$ is defined similarly.

For $k, l \in \mathbb{N}$, let

$$V^0_{kl}(x) = \inf_{\beta \in \Gamma^0_{kl}} \sup_{Y \in M^0_k} J^x(Y, \beta), \tag{2.7}$$

$$U^0_{kl}(x) = \sup_{\alpha \in \Gamma^0_{lk}} \inf_{Z \in M^0_l} J^x(\alpha, Z). \tag{2.8}$$

The following shows that the two formulations are equivalent.

LEMMA 2.2. *For every k, l, $V^0_{kl} = V_{kl}$ and $U^0_{kl} = U_{kl}$.*

PROOF. We only show the claim regarding V_{kl}. Let $\beta \in \Gamma_l$. Define $\beta^0 \in \Gamma^0_{kl}$ by letting, for every $Y \in M^0_k$, $\beta^0[Y]$ be the process component of the pair $\beta[(Y, k)]$. Clearly, for every $Y \in M^0_k$, $J^x((Y, k), \beta) = J^x(Y, \beta^0)$, whence $\sup_{Y \in M_k} J^x(Y, \beta) \geq \sup_{Y \in M^0_k} J^x(Y, \beta^0)$, and $V_{kl}(x) \geq V^0_{kl}(x)$.

Next, let $\beta^0 \in \Gamma^0_{kl}$. Define $\beta : M \to M_l$ as follows. Given $Y \equiv (Y^0, K) \equiv (A, C, K) \in M$, let $Y^k = (A, C \wedge k)$, and set $\beta[Y] = (\beta^0[Y^k], l)$. Note that if, for some K, Y^0 and $\widetilde{Y}^0$ are elements of M^0_K and $Y^0(s) = \widetilde{Y}^0(s)$ on $[0, t]$ then $Y^k(s) = \widetilde{Y}^k(s)$ on $[0, t]$ and so $\beta^0[Y^k]_s = \beta^0[\widetilde{Y}^k]_s$ on $[0, t]$. By definition of β, it follows that $\beta \in \Gamma_l$. Also, if $(Y^0, K) \in M_k$ then $K \leq k$ and thus $J^x((Y^0, K), \beta) = J^x(Y^0, \beta^0)$. This shows that $\sup_{Y \in M_k} J^x(Y, \beta) \leq \sup_{Y^0 \in M^0_k} J^x(Y^0, \beta^0)$. Consequently, $V_{kl}(x) \leq V^0_{kl}(x)$. $\square$

Denote $V_n = V_{k_n l_n}$ and $U_n = U_{k_n l_n}$. The following result is proved in Section 3.

THEOREM 2.1. *For some $n_0 \in \mathbb{N}$, the family $\{V_n; n \geq n_0\}$ is equicontinuous, and so is the family $\{U_n; n \geq n_0\}$.*

Consider the Bellman–Isaacs equations for the upper and, respectively, lower values of the game with bounded controls, namely

$$\begin{cases} \Lambda^+_n(Du, D^2u) - h = 0, & \text{in } G, \\ u = g, & \text{on } \partial G, \end{cases} \tag{2.9}$$

$$\begin{cases} \Lambda^-_n(Du, D^2u) - h = 0, & \text{in } G, \\ u = g, & \text{on } \partial G. \end{cases} \tag{2.10}$$

Solutions to these equations are defined analogously to Definition 1.2, with $\Lambda^\pm_n$ replacing Λ, and where there is no restriction on the derivatives of the test function, that is, $\mathcal{D}$ is replaced with $\mathbb{R}^m \times \mathcal{S}(m)$.

LEMMA 2.3. *There exists $n_1 \in \mathbb{N}$ such that for each $n \geq n_1$, U_n is the unique solution to* (2.9), *and V_n is the unique solution to* (2.10).

PROOF. This follows from a more general result, Theorem 4.1 in Section 4. $\square$

LEMMA 2.4. *Any subsequential uniform limit of U_n or V_n is a solution of* (1.1).

PROOF. Denote by U_0 (resp., V_0) a subsequential limit of U_n $[V_n]$. By relabeling, we assume without loss that U_n (resp., V_n) converges to U_0 $[V_0]$. We will show that U_0 and V_0 are subsolutions of (1.1). The proof that these are supersolutions is parallel.

We start with the proof that U_0 is a subsolution. Fix $x_0 \in G$. Let $\varphi \in \mathcal{C}^2(G)$ be such that $U_0 - \varphi$ is strictly maximized at x_0. Assume first that $D\varphi(x_0) \neq 0$. Since $U_n \to U_0$ uniformly, we can find $\{x_n\} \subset G$, $x_n \to x_0$, where x_n is a local maximum of $U_n - \varphi$ for $n \geq N$. We take N to be larger than n_1 of Lemma 2.3. Since by Lemma 2.3 U_n is a subsolution of (2.9), we have that for $n \geq N$

$$\Lambda_n^+(D\varphi(x_n), D^2\varphi(x_n)) - h(x_n) \leq 0.$$

Thus, by Lemma 2.1,

$$\Lambda(D\varphi(x_0), D^2\varphi(x_0)) - h(x_0) \leq 0$$

as required.

Next, assume that $D\varphi(x_0) = 0$ and $D^2\varphi(x_0) = \lambda I_m$ for some $\lambda \in \mathbb{R}$. In particular, $\varphi(x) = \varphi(x_0) + \frac{\lambda}{2}|x - x_0|^2 + o(|x - x_0|^2)$. We need to show that

$$-2\lambda - h(x_0) \leq 0. \tag{2.11}$$

Consider the case $\lambda \geq 0$. Fix $\delta > 0$ and let $\psi_\delta(x) = \frac{\lambda+\delta}{2}|x - x_0|^2$. Then $U_0 - \psi_\delta$ has a strict maximum at x_0. Since $U_n \to U_0$ uniformly, we can find $\{x_n\} \subset G$, $x_n \to x_0$, where x_n is a local maximum of $U_n - \psi_\delta$. To prove (2.11), it suffices to show that for each $\varepsilon > 0$,

$$-2(\lambda + \delta) - \sup_{x \in \mathbb{B}_\varepsilon(x_0)} h(x) \leq 0. \tag{2.12}$$

To prove (2.12), argue by contradiction and assume that it fails. Then there exists $\varepsilon > 0$ such that

$$-2(\lambda + \delta) - \sup_{x \in \mathbb{B}_\varepsilon(x_0)} h(x) > 0. \tag{2.13}$$

Let $N \geq n_1$ be such that $|x_n - x_0| < \varepsilon$ for all $n \geq N$. Since U_n is a subsolution of (2.9),

$$\mu_n := \Lambda_n^+(D\psi_\delta(x_n), D^2\psi_\delta(x_n)) \leq h(x_n). \tag{2.14}$$

Also,

$$\begin{aligned}\mu_n = \max_{|b|=1,0\le d\le l_n}\min_{|a|=1,0\le c\le k_n}\biggl[-\frac{1}{2}(\lambda+\delta)|a-b|^2 \\ -(\lambda+\delta)(c+d)(a+b)\cdot(x_n-x_0)\biggr] \\ \ge \min_{|a|=1,0\le c\le k_n}\biggl[-\frac{1}{2}(\lambda+\delta)|a-b_n|^2\biggr] = -2(\lambda+\delta),\end{aligned}\tag{2.15}$$

where $b_n = -(x_n - x_0)/|x_n - x_0|$ if $x_n \neq x_0$ and arbitrary otherwise. Thus by (2.13),

$$\mu_n > h(x_n).\tag{2.16}$$

However, this contradicts (2.14). Hence, (2.12) holds and so (2.11) follows.

Consider now the case $\lambda < 0$. Let $\delta > 0$ be such that $\lambda + \delta < 0$. Let ψ_δ be as above. Then $U_0 - \psi_\delta$ has a strict maximum at x_0. Fix $\varepsilon > 0$. Then one can find $\gamma < \varepsilon$ such that

$$U_0(x_0) = U_0(x_0) - \psi_\delta(x_0) > U_0(x) - \psi_\delta(x) \qquad \forall 0 < |x - x_0| \le \gamma.\tag{2.17}$$

Thus, one can find $\eta \in \mathbb{R}^m$ such that $0 < |\eta| < \gamma$ and

$$U_0(x_0) > U_0(x) - \psi_\delta(x+\eta) \qquad \forall x \in \partial\mathbb{B}_\gamma(x_0).\tag{2.18}$$

Let $\psi_{\delta,\eta}(x) = \psi_\delta(x+\eta)$. Let $x_\eta \in \overline{\mathbb{B}_\gamma(x_0)}$ be a maximum point for $U_0 - \psi_{\delta,\eta}$ over $\overline{\mathbb{B}_\gamma(x_0)}$. We claim that

$$x_\eta \notin \partial\mathbb{B}_\gamma(x_0) \quad \text{and} \quad x_\eta \neq x_0 - \eta.\tag{2.19}$$

Suppose the claim holds. Then $D\psi_{\delta,\eta}(x_\eta) \neq 0$, and so from the first part of the proof

$$-2(\lambda+\delta) - h(x_\eta) = \Lambda(D\psi_{\delta,\eta}(x_\eta), D^2\psi_{\delta,\eta}(x_\eta)) - h(x_\eta) \le 0.$$

Since $|x_\eta - x_0| \le \gamma < \varepsilon$, sending $\varepsilon \to 0$ and then $\delta \to 0$ yields (2.11).

We now prove (2.19). From (2.18) and the fact that $\lambda + \delta < 0$,

$$\sup_{x\in\partial\mathbb{B}_\gamma(x_0)} [U_0(x) - \psi_{\delta,\eta}(x)] < U_0(x_0) \le U_0(x_0) - \psi_{\delta,\eta}(x_0).$$

Hence, $x_\eta \notin \partial\mathbb{B}_\gamma(x_0)$. Also

$$\begin{aligned}U_0(x_0-\eta) - \psi_{\delta,\eta}(x_0-\eta) = U_0(x_0-\eta) < U_0(x_0) + \psi_\delta(x_0-\eta) \\ \le U_0(x_0) - \psi_{\delta,\eta}(x_0),\end{aligned}$$

where we used (2.17) and the negativity of the functions ψ_δ and $\psi_{\delta,\eta}$. This shows that $x_\eta \neq x_0 - \eta$, and (2.19) follows. This completes the proof that U_0 is a subsolution of (1.1).

Finally, the argument for V_0 differs only at one point. If we had (V_n, Λ_n^-) instead of (U_n, Λ_n^+), then instead of (2.15), we could write

$$\begin{aligned}\mu_n &= \min_{|a|=1, 0\le c\le k_n} \max_{|b|=1,0\le d\le l_n} \biggl[-\frac{1}{2}(\lambda+\delta)|a-b|^2 \\ &\qquad\qquad - (\lambda+\delta)(c+d)(a+b)\cdot(x_n-x_0)\biggr] \\ &= \max_{|b|=1,0\le d\le l_n}\biggl[-\frac{1}{2}(\lambda+\delta)|a_n-b|^2-(\lambda+\delta)(c_n+d)(a_n+b)\cdot(x_n-x_0)\biggr],\end{aligned}$$

where (a_n, c_n) achieves the minimum, and then by choosing $(b, d) = (-a_n, 0)$,

$$\mu_n \ge -2(\lambda+\delta).$$

Hence, (2.16) is still true. Rest of the argument for the subsolution property of V_0 follows as that for U_0. $\square$

LEMMA 2.5. *Fix $x \in \overline{G}$.*

(i) *One can choose (k_n, l_n) in such a way that* $\limsup_{n\to\infty} V_n(x) \le V(x)$.
(ii) *One can choose (k_n, l_n) in such a way that* $\liminf_{n\to\infty} V_n(x) \ge V(x)$.
Similar statements hold for $U_n(x)$ and $U(x)$.

PROOF. We prove (i) and (ii). The statements regarding $U_n(x)$ and $U(x)$ are proved analogously.

(i) Fix k. Since $\Gamma = \bigcup_{l\ge1}\Gamma_l$, we have that given ε,

$$\begin{aligned}V(x) &\ge \inf_{\beta\in\Gamma}\sup_{Y\in M_k} J^x(Y,\beta) \\ &\ge \inf_{\beta\in\Gamma_l}\sup_{Y\in M_k} J^x(Y,\beta)-\varepsilon \\ &= V_{kl}(x)-\varepsilon,\end{aligned}$$

for all l sufficiently large. This shows $V(x) \ge \limsup_{l\to\infty} V_{kl}(x)$, and (i) follows.

(ii) Fix ε. For each $(k,l)\in\mathbb{N}^2$, let $\beta_{kl}\in\Gamma_l$ be such that

$$\sup_{Y\in M_k} J^x(Y,\beta_{kl}) \le \inf_{\beta\in\Gamma_l}\sup_{Y\in M_k} J^x(Y,\beta)+\varepsilon. \tag{2.20}$$

Fix l. Let β_l be defined by

$$\beta_l[Y] = \beta_{kl}[Y], \qquad Y\in M_k\setminus M_{k-1}, k\in\mathbb{N},$$

where we define M_0 to be the empty set. Then $\beta_l\in\Gamma_l$. Since $M=\bigcup_{k\ge1} M_k$, we have that the following holds provided that k is sufficiently large

$$\begin{aligned} V(x) &\le \sup_{Y\in M} J^x(Y,\beta_l) \\ &\le \sup_{Y\in M_k} J^x(Y,\beta_l)+\varepsilon \\ &= \max_{j\le k} \sup_{Y\in M_j\setminus M_{j-1}} J^x(Y,\beta_{jl})+\varepsilon \\ &\le \inf_{\beta\in\Gamma_l}\sup_{Y\in M_k} J^x(Y,\beta)+2\varepsilon, \end{aligned}$$

where the last inequality follows from (2.20). This shows that, for every l, $V(x) \le \liminf_k V_{kl}(x)$. The result follows. □

PROOF OF THEOREM 1.2. The statement that U and V are solutions of (1.1) follows from Theorem 2.1, Lemmas 2.3, 2.4 and uniqueness of solutions of (1.1), established in [10]. The latter result also yields $U = V$. □

3. Equicontinuity. In this section, we prove Theorem 2.1. With an eye toward estimates needed in Section 4 we will consider a somewhat more general setting. Thanks to Lemma 2.2 we may, and will use the value functions (2.7), (2.8), defined using simple controls and strategies (Definition 2.1). Given X defined as in (1.6) for some $Y, Z \in M^0$, we let for $\gamma \in [0,1)$, $X^\gamma = X + \gamma\widetilde{W}$. Define τ^γ and J_γ as below (1.7) but with X replaced with X^γ. Also denote by U^γ_{kl} and V^γ_{kl} the expressions in (2.7), (2.8) with J replaced with J_γ. We write $U^\gamma_n = U^\gamma_{k_n l_n}$, $V^\gamma_n = V^\gamma_{k_n l_n}$. Theorem 2.1 is an immediate consequence of the following more general result.

THEOREM 3.1. *For some $n_2 \in \mathbb{N}$, the family $\{V^\gamma_n, U^\gamma_n; n \ge n_2, \gamma \in [0,1)\}$ is equicontinuous.*

In what follows, we will suppress γ from the notation unless there is a scope for confusion. We start by showing that the value functions are uniformly bounded. To this end, fix $a^0 \in S^{m-1}$, and note that the constant process $Y^0 := (a^0, 1)$ is in M^0.

LEMMA 3.1. *There exists a constant $c_1 < \infty$ such that*

$$\mathbf{E}[\tau(x, Y^0, Z)^2] \le c_1, \qquad x \in \overline{G}, Z \in M^0, \gamma \in [0,1).$$

PROOF. We only present the proof for the case $\gamma = 0$. The general case follows upon minor modifications. Denote by m_0 the diameter of G. Fix $T > m_0$. By (1.6), with $\alpha_t = a^0 \cdot B_t$, on the event $\tau > T$ one has

$$\int_0^T (1-\alpha_s)\,dW_s + \int_0^T (1+\alpha_s)\,ds \le a^0 \cdot (X_T - X_0) < m_0.$$

Consider the $\{\mathcal{F}_t\}$-martingale, $M_t = \int_0^t (1-\alpha_s)\,dW_s$, with $\langle M\rangle_t = \int_0^t (1-\alpha_s)^2\,ds$. On the event $\langle M\rangle_T < T$,

$$\int_0^T (1+\alpha_s)\,ds = 2T - \int_0^T (1-\alpha_s)\,ds \geq T.$$

So on the set $\{\langle M\rangle_T < T; \tau > T\}$ we have $|M_T| \geq T - m_0$. Letting $\sigma = \inf\{s : \langle M\rangle_s \geq T\}$,

$$\begin{aligned} \mathbf{P}(\tau > T; \langle M\rangle_T < T) &\leq \mathbf{P}(|M_{T\wedge\sigma}| \geq T - m_0) \\ &\leq \frac{m_1 \mathbf{E}\langle M\rangle^2_{T\wedge\sigma}}{(T-m_0)^4} \leq \frac{m_1 T^2}{(T-m_0)^4}. \end{aligned} \tag{3.1}$$

We now consider the event $\{\tau > T; \langle M\rangle_T \geq T\}$. One can find $m_2, m_3 \in (0,\infty)$ such that for all nondecreasing, nonnegative processes $\{\widehat{\gamma}_t\}$,

$$\mathbf{P}\big(H_s + \widehat{\gamma}_s \in (-m_0, m_0); 0 \leq s \leq T\big) \leq m_2 e^{-m_3 T}, \tag{3.2}$$

where H is a one-dimensional Brownian motion. Letting $\gamma_t = \int_0^t (1+D_s)(1+\alpha_s)\,ds$, where $Z = (B, D)$, we see that

$$\{\tau > T; \langle M\rangle_T \geq T\} \subset \{M_s + \gamma_s \in (-m_0, m_0), 0 \leq s \leq T; \langle M\rangle_T \geq T\}.$$

For $u \geq 0$, let $S_u = \inf\{s : \langle M\rangle_s > u\}$. Then, with $\widehat{\gamma}_s = \gamma_{S_s}$,

$$\mathbf{P}(\tau > T; \langle M\rangle_T \geq T) \leq \mathbf{P}\big(H_s + \widehat{\gamma}_s \in (-m_0, m_0); 0 \leq s \leq T\big) \leq m_2 e^{-m_3 T},$$

where the last inequality follows from (3.2). The result now follows on combining the above display with (3.1) $\square$

The inequality $J(x, Y^0, Z) \leq |h|_\infty \mathbf{E}(\tau(x, Y^0, Z)) + |g|_\infty$, where $|h|_\infty = \sup_x |h(x)|$ and $|g|_\infty = \sup_x |g(x)|$, immediately implies the following.

COROLLARY 3.1. *There exists a constant* $c_2 < \infty$ *such that* $|V_n^\gamma(x)| \vee |U_n^\gamma(x)| \leq c_2$, *for all* $x \in \overline{G}$, $\gamma \in [0,1)$ *and* $n \in \mathbb{N}$.

The idea of the proof of equicontinuity, explained in a heuristic manner, is as follows. Let x_1 and x_2 be in G, let $\varepsilon = |x_1 - x_2|$, and let $\delta > 0$. Consider the game with bounded controls for which V_n is the lower value function, for some $n \in \mathbb{N}$. Let the minimizing player select a strategy β^n that is δ-optimal for the initial position x_1; namely $\sup_{Y\in M^0_{k_n}} J^{x_1}(Y, \beta^n) \leq V_n(x_1) + \delta$. Denote the exit time by $\tau_1 = \tau^{x_1}(Y, \beta^n)$ and the exit position by $\xi_1 = X^{x_1}(\tau_1)$. Now, modify the strategy is such a way that the resulting control $Z = \beta^n[Y]$ is only affected for times $t \geq \tau_1$. This way, the payoff incurred remains unchanged. Thus, denoting the modified strategy by $\widetilde{\beta}^n$, we have, for every $Y \in M^0_{k_n}$,

$$J^{x_1}(Y, \widetilde{\beta}^n) \leq V_n(x_1) + \delta.$$

Given a point ξ_2 located inside G, ε away from ξ_1, and a new state process which, at time τ_1 is located at ξ_2, the modified strategy attempts to force this process to exit the domain soon after τ_1 and with a small displacement from ξ_2 (provided that ε is small).

Let now the maximizing player select a control Y^n that is δ-optimal for playing against $\widetilde{\beta}^n$, when starting from x_2. This control is modified after the exit time $\tau^{x_2}(Y^n, \widetilde{\beta}^n)$ in a similar manner to the above. Denoting the modified control by $\widetilde{Y}^n$, we have

$$V_n(x_2) \le J^{x_2}(\widetilde{Y}^n, \widetilde{\beta}^n) + \delta.$$

Hence, $V_n(x_2) - V_n(x_1) \le J^{x_2}(\widetilde{Y}^n, \widetilde{\beta}^n) - J^{x_1}(\widetilde{Y}^n, \widetilde{\beta}^n) + 2\delta$. One can thus estimate the modulus of continuity of V_n by analyzing the payoff incurred when $(\widetilde{Y}^n, \widetilde{\beta}^n)$ is played, considering simultaneously two state processes, starting from x_1 and x_2. The form (1.6) of the dynamics ensures that the processes remain at relative position $x_1 - x_2$ until, at time σ, one of them leaves the domain. The difference between the running payoffs incurred up to that time can be estimated in terms of ε, the modulus of continuity of h, and the expectation of σ. It is not hard to see that the latter is uniformly bounded, owing to Corollary 3.1 and the boundedness of h away from zero. By construction, one of the players will now attempt to force the state process that is still in G to exit. If one can ensure that exit occurs soon after σ and with a small displacement (uniformly in n), then the running payoff incurred between time σ and the exit time is small, and the difference between the terminal payoffs is bounded in terms of ε and the modulus of continuity of g, resulting in an estimate that is uniform in n.

This argument is made precise in the proof of the theorem. Lemmas 3.2 and 3.3 provide the main tools for showing that starting at a state near the boundary, each player may force exit within a short time and with a small displacement. To state these lemmas, we first need to introduce some notation.

We have assumed that G is a bounded C^2 domain in $\mathbb{R}^m$. Thus, there exist $\overline{\rho} \in (0, \frac{1}{8})$, $k \in \mathbb{N}$, $z_j \in \partial G$, $E_j \in \mathcal{O}(m)$, $\xi_j \in C^2(\mathbb{R}^{m-1})$, $j = 1, \dots, k$, such that, with $\mathbb{B}_j = \mathbb{B}_{\overline{\rho}}(z_j)$, $j = 1, \dots, k$, one has $\partial G \subset \bigcup_{j=1}^k \mathbb{B}_j$, and

$$G \cap \mathbb{B}_j = \{E_j y : y_1 > \xi_j(y_2, \dots, y_m)\} \cap \mathbb{B}_j, \qquad j = 1, \dots, k.$$

Here, $\mathcal{O}(m)$ is the space of $m \times m$ orthonormal matrices. Define for $j = 1, \dots, k$, $\widetilde{\varphi}_j : \mathbb{R}^m \to \mathbb{R}$ as

$$\widetilde{\varphi}_j(y) = y_1 - \xi_j(y_2, \dots, y_m), \qquad y \in \mathbb{R}^m.$$

Let $\varphi_j(x) = \widetilde{\varphi}_j(E_j^{-1} x)$, $x \in \mathbb{R}^m$. Then $|D\varphi_j(x)| \ge 1$, $x \in \mathbb{R}^m$, $j = 1, \dots, k$. Furthermore,

$$G \cap \mathbb{B}_j = \{x : \varphi_j(x) > 0\} \cap \mathbb{B}_j, \qquad j = 1, \dots, k.$$

Let $0 < \rho_0 < \overline{\rho}$ be such that $\partial G \subset \bigcup_{j=1}^k \mathbb{B}_{\rho_0}(z_j)$. For $\varepsilon > 0$, denote

$$\mathbf{X}_\varepsilon = \{(x_1, x_2) : x_1 \in \partial G, x_2 \in G, |x_1 - x_2| \le \varepsilon\}.$$

Let $\underline{j} : \partial G \to \{1, \dots, k\}$ be a measurable map with the property

$$x \in \mathbb{B}_{\rho_0}(z_{\underline{j}(x)}) \qquad \text{for all } x \in \partial G.$$

For existence of such a map see, for example, Theorem 10.1 of [6]. Then, for every $\varepsilon \le \rho_1 := \frac{\overline{\rho}-\rho_0}{4}$,

$$(3.3) \qquad (x_1, x_2) \in \mathbf{X}_\varepsilon \qquad \text{implies } \overline{\mathbb{B}_{\rho_1}(x_i)} \subset \mathbb{B}_{\underline{j}(x_1)}, \qquad i = 1, 2.$$

For $j = 1, \dots, k$ and $x_0 \in \mathbb{B}_j$, define $\psi_j^{x_0} : \mathbb{R}^m \to \mathbb{R}$ as

$$\psi_j^{x_0}(x) = \varphi_j(x) + |x - x_0|^2.$$

Also, note that $|D\psi_j^{x_0}| \ge \frac{1}{2}$ in $\mathbb{B}_j$. Define $\pi_j^{x_0} : \mathbb{R}^m \to \mathcal{S}^{m-1}$ such that it is Lipschitz, and

$$(3.4) \qquad \pi_j^{x_0}(x) = -\frac{D\psi_j^{x_0}(x)}{|D\psi_j^{x_0}(x)|}, \qquad x \in \mathbb{B}_j.$$

Given a strategy β, and a point x_2, we seek a control $Y = (A, C)$ that forces a state process starting from x_2 to exit in a short time and with a small displacement from x_2 (provided that x_2 is close to the boundary). We would like to determine Y via the functions π_j just constructed, in such a way that the following relation holds:

$$(3.5) \qquad A(t) = \pi(X(t)), \qquad C(t) = c^0,$$

where $\pi = \pi_{\underline{j}(x_1)}^{x_2}$ and $c^0 > 0$ is some constant. Making A be oriented in the negative direction of the gradient of φ_j allows us to show that the state is "pushed" toward the boundary. The inclusion of a quadratic term in $\psi_j^{x_0}$ ensures in addition that the sublevel sets $\{\psi_j < a\}$ are contained in a small vicinity of x_0, provided smallness of a and $\mathrm{dist}(x_0, \partial G)$. The latter property enables us to show that the process does not wander a long way along the boundary before exiting.

The difficulty we encounter is that due to the feedback nature of A in (3.5) we cannot ensure (local) existence of solutions of the set of (1.6) and (3.5). Take, for example, a strategy β that is given as $\beta[Y]_t = b(Y_t)$, $Y \in M^0$, where b is some measurable map from $\mathcal{H}$ to $\mathcal{H}$. Along with (1.6), and (3.5) this defines X as a solution to an SDE with general measurable coefficients. However, as is well known, the SDE may not admit any solution in this generality. To overcome this problem, we will construct a Y that approximates the Y we seek in (3.5) via a time discretization.

Let $\mathbf{\Sigma}_\varepsilon$ denote the collection of all quintuples $\boldsymbol{\sigma} = (\sigma, \xi_1, \xi_2, \beta, Y)$ such that σ is an a.s. finite $\mathcal{F}_t$ stopping time, ξ_1 and ξ_2 are $\mathcal{F}_\sigma$-measurable random variables satisfying $(\xi_1, \xi_2) \in \mathbf{X}_\varepsilon$ a.s., $\beta \in \Gamma^0$, and $Y \in M^0$. Fix $\gamma \in [0, 1)$.

Let $\boldsymbol{\sigma} = (\sigma, \xi_1, \xi_2, \beta, Y) \in \mathbf{\Sigma}_{\rho_1}$ be given. Let $j^*(\omega) = \underline{j}(\xi_1(\omega))$. Denote

$$\Phi \equiv \Phi(\omega, \cdot) = \varphi_{j^*(\omega)}, \qquad \Psi \equiv \Psi(\omega, \cdot) = \psi_{j^*(\omega)}^{\xi_2(\omega)}, \qquad \Pi \equiv \Pi(\omega, \cdot) = \pi_{j^*(\omega)}^{\xi_2(\omega)}.$$

We define a sequence of processes $(X^{(i)}, Y^{(i)})_{i\ge 0}$ as follows. Let $Y_t^{(0)} \equiv (A_t^{(0)}, C_t^{(0)})$ be given by

$$Y_t^{(0)} = \begin{cases} Y_t, & t < \sigma, \\ (\Pi(\xi_2), c^0), & t \ge \sigma. \end{cases}$$

The constant c^0 above will be chosen later. Denote $(B^{(0)}, D^{(0)}) = \beta[Y^{(0)}]$. Define, for $t \ge \sigma$,

$$X_t^{(0)} = \xi_2 + \int \mathbf{1}_{[\sigma,t]}(s)\big([A_s^{(0)} - B_s^{(0)}]\,dW_s + \gamma\,d\widetilde{W}_s\big) + \int_\sigma^t [C_s^{(0)} + D_s^{(0)}][A_s^{(0)} + B_s^{(0)}]\,ds.$$

The process $X^{(0)}$ can be defined arbitrarily for $t < \sigma$. Set $\eta_0 = \sigma$. We now define recursively, for all $i \ge 1$,

$$\eta_i = (\eta_{i-1} + \varepsilon) \wedge \inf\left\{t \ge \eta_{i-1} : \big|X_t^{(i-1)} - X_{\eta_{i-1}}^{(i-1)}\big| \ge \varepsilon \text{ or } \int_{\eta_{i-1}}^t R_s^{(i)}\,ds \ge t - \eta_{i-1}\right\}, \tag{3.6}$$

where $R_s^{(i)} = [C_s^{(i-1)} + D_s^{(i-1)}]D\Psi(X_s^{(i-1)}) \cdot [A_s^{(i-1)} - \Pi(X_s^{(i-1)})]$,

$$X_t^{(i)} = X_t^{(i-1)}, \qquad Y_t^{(i)} = Y_t^{(i-1)}, \qquad 0 \le t < \eta_i,$$

$$Y_t^{(i)} \equiv \big(A_t^{(i)}, C_t^{(i)}\big) = \big(\Pi\big(X_{\eta_i}^{(i-1)}\big), c^0\big), \qquad t \ge \eta_i,$$

$$\big(B^{(i)}, D^{(i)}\big) = \beta\big[Y^{(i)}\big], \tag{3.7}$$

$$X_t^{(i)} = X_{\eta_i}^{(i-1)} + \int \mathbf{1}_{[\eta_i,t]}(s)\big([A_s^{(i)} - B_s^{(i)}]\,dW_s + \gamma\,d\widetilde{W}_s\big) + \int_{\eta_i}^t [A_s^{(i)} + B_s^{(i)}][C_s^{(i)} + D_s^{(i)}]\,ds, \qquad t > \eta_i.$$

It is easy to check that $\eta_0 < \eta_1 < \cdots$ and $\eta_i \to \infty$ a.s. Define $X_t = X_t^{(i)}$, $Y_t = Y_t^{(i)}$ if $t \le \eta_i$. Let $\rho = \rho_1^2$ and

$$\begin{aligned} \tau_\rho &= \inf\{t \ge \sigma : \Psi(X_t) \ge \rho\}, \\ \tau_G &= \inf\{t \ge \sigma : X_t \in G^c\}, \\ \tau &= \tau_\rho \wedge \tau_G. \end{aligned} \tag{3.8}$$

Define

$$\overline{Y}_t \equiv (\overline{A}_t, \overline{C}_t) = \begin{cases} Y_t, & t < \tau, \\ (a^0, c^0), & t \ge \tau, \end{cases}$$

$$(\overline{B}, \overline{D}) = \beta(\overline{Y}),$$

where a^0 is as fixed at the beginning of the section. Let

$$\overline{X}_t = \xi_2 + \int \mathbf{1}_{[\sigma,t]}(s)([\overline{A}_s - \overline{B}_s]\,dW_s + \gamma\,d\widetilde{W}_s) + \int_\sigma^t [\overline{C}_s + \overline{D}_s][\overline{A}_s + \overline{B}_s]\,ds.$$

We write

$$\overline{X} = \overline{X}[\sigma, \xi_1, \xi_2, \beta, Y], \qquad \overline{Y} = \overline{Y}[\sigma, \xi_1, \xi_2, \beta, Y].$$

Note that if $\overline{\tau}_\rho$ and $\overline{\tau}_G$ are defined by (3.8) upon replacing X by $\overline{X}$ then $\overline{\tau} := \overline{\tau}_\rho \wedge \overline{\tau}_G = \tau_\rho \wedge \tau_G = \tau$, because $\overline{X}$ differs from X only after time τ. We write $\overline{\tau} = \overline{\tau}[\sigma, \xi_1, \xi_2, \beta, Y]$. Similar notation will be used for $\overline{\tau}_\rho$ and $\overline{\tau}_G$.

LEMMA 3.2. *There exists a $c^0 \in (0,\infty)$ and a modulus ϑ such that for every $\varepsilon \in (0, \rho_1)$, and $\gamma \in [0, 1)$, if $\boldsymbol{\sigma} = (\sigma, \xi_1, \xi_2, \beta, Y) \in \boldsymbol{\Sigma}_\varepsilon$ and $\overline{\tau}_G = \overline{\tau}_G[\boldsymbol{\sigma}]$, one has*:

(i) $\mathbf{E}\{\overline{\tau}_G - \sigma | \mathcal{F}_\sigma\} \le \vartheta(\varepsilon)$,
(ii) $\mathbf{E}\{|\overline{X} - \xi_2|^2_{*,\overline{\tau}_G} | \mathcal{F}_\sigma\} \le \vartheta(\varepsilon)$, *where* $|\overline{X} - \xi_2|_{*,\overline{\tau}_G} = \sup_{t\in[\sigma,\overline{\tau}_G]} |\overline{X}(t) - \xi_2|$.

Proof of the lemma is provided after the proof of Theorem 3.1.

Next, we construct a strategy $\beta^* \in \Gamma^0$ with analogous properties. Here, existence of solutions is not an issue, and discretization is not needed.

Fix $(x_1, x_2) \in \mathbf{X}_{\rho_1}$. Let $j = \underline{j}(x_1)$, $\widetilde{\Psi} = \psi_j^{x_2}$, and $\widetilde{\Pi} = \pi_j^{x_2}$. Given $Y = (A, C) \in M^0$, let $\widetilde{X}$ solve

$$\widetilde{X}_t = x_2 + \int_0^t ([A_s - \widetilde{\Pi}(\widetilde{X}_s)]\,dW_s + \gamma\,d\widetilde{W}_s) + \int_0^t [C_s + d^0][A_s + \widetilde{\Pi}(\widetilde{X}_s)]\,ds,$$

where d^0 is a constant to be determined later. Let

$$\begin{aligned} \widetilde{\tau}_\rho &= \inf\{t : \widetilde{\Psi}(\widetilde{X}_t) \ge \rho\}, \\ \widetilde{\tau}_G &= \inf\{t : \widetilde{X}_t \in G^c\}, \\ \widetilde{\tau} &= \widetilde{\tau}_\rho \wedge \widetilde{\tau}_G. \end{aligned} \tag{3.9}$$

Define $Z^* \in M^0$ as

$$Z_s^* \equiv (B_s^*, D_s^*) = \begin{cases} (\widetilde{\Pi}(\widetilde{X}_s), d^0), & s < \widetilde{\tau}, \\ (a^0, d^0), & s \ge \widetilde{\tau}. \end{cases}$$

Note that $\beta^*[Y](s) := \widetilde{Z}_s$, $s \ge 0$ defines a strategy. Let

$$X_t^* = x_2 + \int_0^t ([A_s - B_s^*]\,dW_s + \gamma\,d\widetilde{W}_s) + \int_0^t [C_s + d^0][A_s + B_s^*]\,ds.$$

Define τ_ρ^*, τ_G^* and τ^* by replacing $\widetilde{X}$ with X^* in (3.9), and note that $\tau^* = \widetilde{\tau}$. To make the dependence explicit, we write

$$X^* = X^*[x_1, x_2, Y, \overline{W}], \qquad Z^* = Z^*[x_1, x_2, Y, \overline{W}], \qquad \tau^* = \tau^*[x_1, x_2, Y, W].$$

LEMMA 3.3. *There exists* $d^0 \in (0,\infty)$ *and a modulus* $\widetilde{\vartheta}$ *such that for all* $\varepsilon \in (0,\rho_1)$, $Y \in M^0$, $(x_1,x_2) \in \mathbf{X}_\varepsilon$, *if* $\tau_G^* = \tau_G^*[x_1,x_2,Y,W]$, *one has*:

(i) $\mathbf{E}[\tau_G^*] \le \widetilde{\vartheta}(\varepsilon)$,

(ii) $\mathbf{E}[|X^* - x_2|^2_{*,\tau_G^*}] \le \widetilde{\vartheta}(\varepsilon)$, *where* $|X^* - x_2|_{*,\tau_G^*} = \sup_{t\in[0,\tau_G^*]}|X^*(t) - x_2|$.

The proof of Lemma 3.3 is very similar to (in fact somewhat simpler than) the proof of Lemma 3.2, and therefore will be omitted.

If σ is an a.s. finite $\{\mathcal{F}_t\}$-stopping time and (ξ_1,ξ_2) are $\mathcal{F}_\sigma$-measurable random variables such that $(\xi_1,\xi_2) \in \mathbf{X}_{\rho_1}$ a.s., then we define the $\mathcal{G}_t = \mathcal{F}_{t+\sigma}$ adapted processes

$$\overline{X}_t^* = X^*[\xi_1,\xi_2,\widehat{Y}_\sigma,\widehat{W}_\sigma](t),$$
$$\overline{Z}_t^* = Z^*[\xi_1,\xi_2,\widehat{Y}_\sigma,\widehat{W}_\sigma](t),$$

where $\widehat{Y}_\sigma(t) = Y(t+\sigma)$ and $\widehat{W}_\sigma(t) = \overline{W}(t+\sigma) - \overline{W}(\sigma)$, $t \ge 0$. To make the dependence explicit, write

$$\overline{X}^* = \overline{X}^*[\sigma,\xi_1,\xi_2,Y], \qquad \overline{Z}^* = \overline{Z}^*[\sigma,\xi_1,\xi_2,Y].$$

PROOF OF THEOREM 2.1. Fix $x_1, x_2 \in G$ and $\gamma \in [0,1)$. We will suppress γ from the notation. Assume that $|x_1 - x_2| = \varepsilon < \rho_1$, so that Lemmas 3.2 and 3.3 are in force (see Figure 1). Let n_0 be large enough so that $l_n, k_n \ge \max\{c^0, d^0\}$ for all $n \ge n_0$. Given $\delta \in (0,1)$ and $n \ge n_0$, let $\beta^n \in \Gamma^0_{k_n l_n}$ be such that

$$\sup_{Y \in M^0_{k_n}} J^{x_1}(Y,\beta^n) - \delta \le V_n(x_1) \le c_1.$$

For $Y \in M^0$ write $\tau^{1,n}(Y) := \tau^{x_1}(Y,\beta^n)$ and $X^{1,n}(Y) := X^{x_1}(Y,\beta^n)$. Note that

$$\underline{h}\mathbf{E}[\tau^{1,n}(Y)] - |g|_\infty \le c_1 + 1,$$

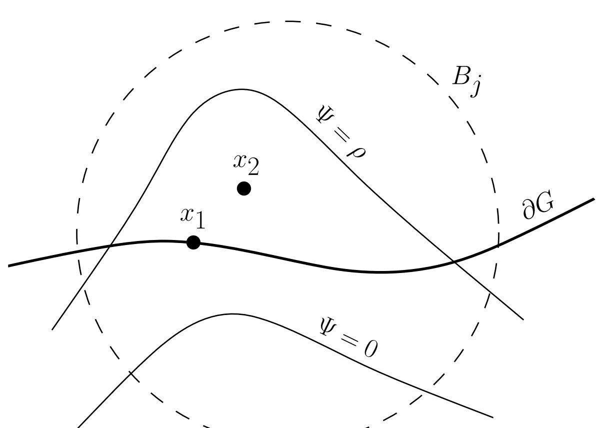

FIG. 1.

hence for every n and every $Y \in M^0_{k_n}$,

$$\mathbf{E}[\tau^{1,n}(Y)] \le m_1, \tag{3.10}$$

where $m_1 < \infty$ is a constant that does not depend on n.

Define $\widetilde{\beta}^n \in \Gamma^0_{k_n l_n}$ as follows. For $Y \in M^0$, let

$$\xi_1^{1,n}(Y) = X^{1,n}_{\tau^{1,n}(Y)}(Y), \qquad \xi_2^{1,n}(Y) = \xi_1^{1,n}(Y) + x_2 - x_1,$$

$$\widetilde{\beta}^n[Y]_t = \begin{cases} \beta^n[Y]_t, & t < \tau^{1,n}(Y), \\ \overline{Z}^*[\tau^{1,n}(Y), \xi_1^{1,n}(Y), \xi_2^{1,n}(Y), Y], & t \ge \tau^{1,n}(Y), \xi_2^{1,n}(Y) \in \overline{G}, \\ \text{arbitrarily defined}, & t \ge \tau^{1,n}(Y), \xi_2^{1,n}(Y) \in \overline{G}^c. \end{cases}$$

Note that for every $Y \in M^0_{k_n}$, $J^{x_1}(Y, \beta^n) = J^{x_1}(Y, \widetilde{\beta}^n)$. Next, choose $Y^n \in M^0_{k_n}$ such that

$$V_n(x_2) \le \sup_{Y \in M^0_{k_n}} J^{x_2}(Y, \widetilde{\beta}^n) \le J^{x_2}(Y^n, \widetilde{\beta}^n) + \delta.$$

Let $\tau^{2,n} = \tau^{x_2}(Y^n, \widetilde{\beta}^n)$, and $X^{2,n} = X^{x_2}(Y^n, \widetilde{\beta}^n)$. Let

$$\xi_2^{2,n} = X^{2,n}_{\tau^{2,n}}, \qquad \xi_1^{2,n} = \xi_2^{2,n} + x_1 - x_2.$$

Define $\widetilde{Y}^n \in M^0_{k_n}$ as

$$\widetilde{Y}^n_t = \begin{cases} Y^n_t, & t < \tau^{2,n}, \\ \overline{Y}[\tau^{2,n}, \xi_2^{2,n}, \xi_1^{2,n}, \widetilde{\beta}^n, Y^n](t), & t \ge \tau^{2,n}, \xi_1^{2,n} \in \overline{G}, \\ \text{arbitrarily defined}, & t \ge \tau^{2,n}, \xi_1^{2,n} \in \overline{G}^c. \end{cases}$$

Note that $J^{x_2}(Y^n, \widetilde{\beta}^n) = J^{x_2}(\widetilde{Y}^n, \widetilde{\beta}^n)$. Thus

$$V_n(x_2) - V_n(x_1) - 2\delta \le J^{x_2}(\widetilde{Y}^n, \widetilde{\beta}^n) - J^{x_1}(\widetilde{Y}^n, \widetilde{\beta}^n). \tag{3.11}$$

For $k = 1, 2$, let

$$\sigma^{k,n} = \tau^{x_k}(\widetilde{Y}^n, \widetilde{\beta}^n), \qquad \widetilde{X}^{k,n} = X^{x_k}(\widetilde{Y}^n, \widetilde{\beta}^n), \qquad \Xi^{k,n} = \widetilde{X}^{k,n}_{\sigma^{k,n}}.$$

For $m_0 \ge 0$, let $\vartheta_g(m_0) = \sup\{|g(x) - g(y)| : x, y \in \partial G, |x - y| \le m_0\}$ and $\vartheta_h(m_0) = \sup\{|h(x) - h(y)| : x, y \in G, |x - y| \le m_0\}$. Using (3.10), the right-hand side of (3.11) can be bounded by

$$\mathbf{E}\vartheta_g(|\Xi^{1,n} - \Xi^{2,n}|) + c_3\vartheta_h(\varepsilon) + |h|_\infty \mathbf{E}[(\sigma^{1,n} \vee \sigma^{2,n}) - (\sigma^{1,n} \wedge \sigma^{2,n})]. \tag{3.12}$$

On the set $\sigma^{1,n} \le \sigma^{2,n}$, we have $|\Xi^{1,n} - \Xi^{2,n}| \le \varepsilon + |\widetilde{X}^{2,n}_{\sigma^{1,n}} - \widetilde{X}^{2,n}_{\sigma^{2,n}}|$. Hence, by Lemma 3.3(ii),

$$\mathbf{E}\big[|\Xi^{1,n} - \Xi^{2,n}|^2 \mathbf{1}_{\{\sigma^{1,n} \le \sigma^{2,n}\}}\big] \le \vartheta_1(\varepsilon)$$

for some modulus ϑ_1. Using Lemma 3.2(ii), a similar estimate holds on the complement set, and consequently, the first term of (3.12) is bounded by $\vartheta_2(\varepsilon)$, for some modulus ϑ_2. By Lemmas 3.2(i) and 3.3(i), the last term of (3.12) is bounded by $|h|_\infty(\vartheta(\varepsilon)+\widetilde{\vartheta}(\varepsilon))$. Hence, $V_n(x_2)-V_n(x_1)\le 2\delta+\vartheta_3(|x_1-x_2|)$ for some modulus ϑ_3, and the equicontinuity of $\{V_n^\gamma; n,\gamma\}$ follows on sending $\delta\to 0$. The proof of equicontinuity of $\{U_n^\gamma; n,\gamma\}$ is similar, and therefore omitted. $\square$

PROOF OF LEMMA 3.2. We will only present the proof for the case $\gamma=0$. The general case follows upon minor modifications. Denote

$$\psi_{1,\infty}=\sup_{x_0,x\in\overline{G}}\sup_j|D\psi_j^{x_0}(x)|,\qquad \psi_{2,\infty}=\sup_{x_0,x\in\overline{G}}\sup_j|D^2\psi_j^{x_0}(x)|,$$

and let $\varphi_{1,\infty}$, $\varphi_{2,\infty}$ be defined analogously. Let $\varepsilon>0$ and $\boldsymbol{\sigma}\equiv(\sigma,\xi_1,\xi_2,\beta,Y)\in\boldsymbol{\Sigma}_\varepsilon$ be given, let $\overline{X}=\overline{X}[\boldsymbol{\sigma}]$, $\overline{Y}=\overline{Y}[\boldsymbol{\sigma}]$, $\overline{\tau}_\rho=\overline{\tau}_\rho[\boldsymbol{\sigma}]$, $\overline{\tau}_G=\overline{\tau}_G[\boldsymbol{\sigma}]$, and $\overline{\tau}=\overline{\tau}[\boldsymbol{\sigma}]$. Let

$$\overline{\tau}_0=\inf\{t\ge\sigma:\Psi(\overline{X}_t)\le 0\},\qquad \overline{\tau}_B=\inf\{t\ge\sigma:\overline{X}_t\notin\mathbb{B}_{j^*}\},$$

where we recall that $j^*=\underline{j}(\xi_1)$. We have $\overline{\tau}\equiv\overline{\tau}_\rho\wedge\overline{\tau}_G\le\overline{\tau}_\rho\wedge\overline{\tau}_0$, because $\Psi\ge\Phi$. Also, by (3.3), $\overline{\tau}\le\overline{\tau}_B$. By Itô's formula, for $t>\sigma$,

$$\begin{aligned}\Psi(\overline{X}_t)=\Phi(\xi_2)&+\int \mathbf{1}_{[\sigma,t]}(s)D\Psi(\overline{X}_s)[\overline{A}_s-\overline{B}_s]\,dW_s\\&+\int_\sigma^t D\Psi(\overline{X}_s)[\overline{A}_s+\overline{B}_s][\overline{C}_s+\overline{D}_s]\,ds\\&+\frac{1}{2}\int_\sigma^t[\overline{A}_s-\overline{B}_s]'D^2\Psi(\overline{X}_s)[\overline{A}_s-\overline{B}_s]\,ds.\end{aligned}$$

For $t\le\overline{\tau}$, using (3.4) and (3.7),

$$\begin{aligned}D\Psi(\overline{X}_s)[\overline{A}_s-\overline{B}_s]&=D\Psi(\overline{X}_s)[\Pi(\overline{X}_s)-\overline{B}_s]+D\Psi(\overline{X}_s)[\overline{A}_s-\Pi(\overline{X}_s)]\\&=-|D\Psi(\overline{X}_s)|(1-\alpha_s-\delta_s),\end{aligned}$$

where $\alpha_s=-|D\Psi(\overline{X}_s)|^{-1}D\Psi(\overline{X}_s)\cdot\overline{B}_s$, and $\delta_s=-\Pi(\overline{X}_s)\cdot[\overline{A}_s-\Pi(\overline{X}_s)]$. Note that $|\alpha_s|\le 1$. Moreover, using the inequality $|\frac{v}{|v|}\cdot(\frac{u}{|u|}-\frac{v}{|v|})|\le 2|v|^{-1}|u-v|$ along with (3.6), recalling the definition of $\overline{A}$ and the fact $|D\Psi(\overline{X}_s)|\ge 1/2$, we see that

$$|\delta_s|\le 4\varepsilon\psi_{2,\infty}.$$

Furthermore,

$$\begin{aligned}&D\Psi(\overline{X}_s)[\overline{A}_s+\overline{B}_s][\overline{C}_s+\overline{D}_s]\\&\quad=D\Psi(\overline{X}_s)[\Pi(\overline{X}_s)+\overline{B}_s][\overline{C}_s+\overline{D}_s]+D\Psi(\overline{X}_s)[\overline{A}_s-\Pi(\overline{X}_s)][\overline{C}_s+\overline{D}_s]\\&\quad=-|D\Psi(\overline{X}_s)|(1+\alpha_s)(c^0+\overline{D}_s)+e_s,\end{aligned}$$

where, by (3.6), for $\sigma\le t_1\le t_2\le\overline{\tau}$

$$\int_{t_1}^{t_2}e_s\,ds\le\varepsilon+t_2-t_1.$$

Finally, we can estimate

$$p_s := \tfrac{1}{2}[\overline{A}_s - \overline{B}_s]' D^2\Psi(\overline{X}_s)[\overline{A}_s - \overline{B}_s]$$

by $|p_s| \le 2\psi_{2,\infty}$. Shifting time by σ, we denote $\mathcal{G}_t = \mathcal{F}_{t+\sigma}$, $\check{W}_t = W_{t+\sigma} - W_\sigma$, and

$$(\check{X}_t, \check{D}_t, \check{\alpha}_t, \check{\delta}_t, \check{e}_t, \check{p}_t) = (\overline{X}_{t+\sigma}, \overline{D}_{t+\sigma}, \alpha_{t+\sigma}, \delta_{t+\sigma}, e_{t+\sigma}, p_{t+\sigma}).$$

Denote also $m_t = |D\Psi(\check{X}_s)|$, let M be the $\mathcal{G}_t$-martingale

$$M_t = -\int_0^t m_s(1 - \check{\alpha}_s - \check{\delta}_s)\, d\check{W}_s$$

and set

$$\begin{aligned} &\mu_t := \langle M\rangle_t = \int_0^t m_s^2(1 - \check{\alpha}_s - \check{\delta}_s)^2\, ds, \\ &P_t = -\int_0^t m_s(1 + \check{\alpha}_s)(c^0 + \check{D}_s)\, ds, \qquad Q_t = \int_0^t (\check{e}_s + \check{p}_s)\, ds, \\ &\Psi_t = \Psi(\check{X}_t). \end{aligned} \tag{3.13}$$

Combining the above estimates, we have for $0 \le s \le t \le \overline{\tau} - \sigma$,

$$\Psi_t = \Psi_0 + M_t + P_t + Q_t, \tag{3.14}$$

$$Q_t - Q_s \le \varepsilon + r(t - s), \tag{3.15}$$

where $r = 2\psi_{2,\infty} + 1$. Note that $m_t \ge 1/2$ for $s \le \overline{\tau}_B - \sigma$, and recall that $\overline{\tau}_B \ge \overline{\tau} \equiv \overline{\tau}_\rho \wedge \overline{\tau}_G$. We have for $t \le \overline{\tau} - \sigma$, assuming without loss of generality $4\varepsilon\psi_{2,\infty} < 1/32$,

$$\begin{aligned} r &= \frac{r}{4}(1 - \check{\alpha}_t - \check{\delta}_t + 1 + \check{\alpha}_t + \check{\delta}_t)^2 \le \frac{r}{2}(1 - \check{\alpha}_t - \check{\delta}_t)^2 + 2r(1 + \check{\alpha}_t + \check{\delta}_t) \\ &\le 2rm_t^2(1 - \check{\alpha}_t - \check{\delta}_t)^2 + 4rm_t(1 + \check{\alpha}_t) + 2r\check{\delta}_t \end{aligned}$$

and

$$\check{\delta}_t \le (1 + \check{\alpha}_t) + \tfrac{1}{8}(1 - \check{\alpha}_t - \check{\delta}_t)^2 \le 2m_t(1 + \check{\alpha}_t) + \tfrac{1}{2}m_t^2(1 - \check{\alpha}_t - \check{\delta}_t)^2.$$

Hence, for $t \le \overline{\tau} - \sigma$, we have $r \le 3rm_t^2(1 - \check{\alpha}_t - \check{\delta}_t)^2 + 8rm_t(1 + \check{\alpha}_t)$. Thus, by (3.14), if c^0 is chosen larger than $8r$, we have

$$\Psi_t = \Psi_0 + M_t + 3r\mu_t + \widetilde{P}_t + \widetilde{Q}_t, \tag{3.16}$$

where

$$\widetilde{P}_t = -\int_0^t m_s(1 + \check{\alpha}_s)(c^0 + \check{D}_s - 8r)\, ds,$$

$\widetilde{Q}_0 = 0$, and

$$\widetilde{P}_t - \widetilde{P}_s \le 0, \qquad \widetilde{Q}_t - \widetilde{Q}_s \le \varepsilon, \qquad 0 \le s \le t \le \overline{\tau} - \sigma. \tag{3.17}$$

We will write $\widehat{\mathbf{P}}$ for $\mathbf{P}[\cdot|\mathcal{G}_0]$, and $\widehat{\mathbf{E}}$ for the respective conditional expectation. The proof will proceed in several steps.

STEP 1. *For some $\nu_1 \in (0, \infty)$,*

$$\sup_{\sigma \in \mathbf{\Sigma}_{\rho_1}} \widehat{\mathbf{E}}[(\overline{\tau} - \sigma)^2] \le \nu_1, \qquad a.s.$$

STEP 2. *For some $\nu_2 \in (0, \infty)$,*

$$\sup_{\sigma \in \mathbf{\Sigma}_{\rho_1}} \widehat{\mathbf{E}}[(\overline{\tau}_G - \sigma)^2] \le \nu_2, \qquad a.s.$$

Note that Step 2 is immediate from Step 1 and Lemma 3.1 because by construction, a constant control is used after time $\overline{\tau}$.

STEP 3. *There exists a modulus ϑ_1 such that*

$$\sup_{\sigma \in \mathbf{\Sigma}_{\varepsilon}} \widehat{\mathbf{P}}[\overline{\tau}_G - \sigma > \vartheta_1(\varepsilon), \overline{\tau}_\rho > \overline{\tau}_G] \le \vartheta_1(\varepsilon), \qquad \varepsilon > 0.$$

STEP 4. *There exists a modulus ϑ_2 such that*

$$\sup_{\sigma \in \mathbf{\Sigma}_{\varepsilon}} \widehat{\mathbf{P}}[\overline{\tau}_\rho \le \overline{\tau}_G] \le \vartheta_2(\varepsilon), \qquad \varepsilon > 0.$$

Based on these steps, part (i) of the lemma is established as follows. Writing E_ε for the event $\overline{\tau}_G - \sigma > \vartheta_1(\varepsilon)$,

$$\begin{aligned}\widehat{\mathbf{E}}[\overline{\tau}_G - \sigma] &= \widehat{\mathbf{E}}[(\overline{\tau}_G - \sigma)\mathbf{1}_{E_\varepsilon}] + \widehat{\mathbf{E}}[(\overline{\tau}_G - \sigma)\mathbf{1}_{E_\varepsilon^c}] \\ &\le [\widehat{\mathbf{E}}[(\overline{\tau}_G - \sigma)^2]\widehat{\mathbf{P}}(E_\varepsilon)]^{1/2} + \vartheta_1(\varepsilon) \\ &\le \nu_2^{1/2}[\vartheta_1(\varepsilon) + \vartheta_2(\varepsilon)]^{1/2} + \vartheta_1(\varepsilon),\end{aligned}$$

where the first inequality uses Cauchy–Schwarz, and the second uses Steps 2, 3 and 4.

To show part (ii) of the lemma, use Steps 3 and 4 to write

$$\begin{aligned}\widehat{\mathbf{E}}[|\overline{X} - \xi_2|^2_{*,\overline{\tau}_G}] &\le \widehat{\mathbf{E}}\big[|\overline{X} - \xi_2|^2_{*,\overline{\tau}_G}\mathbf{1}_{E_\varepsilon^c \cap \{\overline{\tau}_\rho > \overline{\tau}_G\}}\big] \\ &\quad + [\vartheta_1(\varepsilon) + \vartheta_2(\varepsilon)]\,\mathrm{diam}(G)^2.\end{aligned} \tag{3.18}$$

By (3.14) and (3.15), we can estimate

$$\widehat{\mathbf{E}}\Big[\sup_{t \in [\sigma, \overline{\tau}_G]} \Psi(\overline{X}_t)\mathbf{1}_{E_\varepsilon^c \cap \{\overline{\tau}_\rho > \overline{\tau}_G\}}\Big] \le \varphi_{1,\infty}\varepsilon + 3\vartheta_1(\varepsilon)^{1/2}\psi_{1,\varepsilon} + r\vartheta_1(\varepsilon) + \varepsilon. \tag{3.19}$$

Thus, noting that $\Phi(\overline{X}_t) \geq 0$ on $\{\overline{\tau}_\rho > \overline{\tau}_G; t \in [\sigma, \overline{\tau}_G]\}$, we have on this set,

$$\Psi(\overline{X}_t) \geq |\overline{X}_t - \xi_2|^2.$$

Part (ii) of the lemma now follows on using the above inequality and (3.19) in (3.18).

In order to complete the proof, we need to establish the statements in Steps 1, 3 and 4.

PROOF OF STEP 1. Let t be given. Let F_t denote the event $\{\Psi_s \in (0, \rho), 0 \leq s \leq t\}$. We have

$$\begin{aligned} \widehat{\mathbf{P}}(\overline{\tau} - \sigma > t) &= \widehat{\mathbf{P}}(\overline{\tau} - \sigma > t, \overline{\tau}_0 - \sigma > t, \overline{\tau}_B - \sigma > t) \\ &\leq \widehat{\mathbf{P}}(F_t, \overline{\tau}_B - \sigma > t). \end{aligned} \tag{3.20}$$

Denote $S_u = \inf\{s : \mu_s > u\}$, where we recall that the infimum over an empty set is taken to be ∞. Let $\kappa \in (0, 1/16)$. Then

$$\begin{aligned} &\widehat{\mathbf{P}}(F_t, \overline{\tau}_B - \sigma > t, \mu_t > \kappa t) \\ &\quad \leq \widehat{\mathbf{P}}\big(\mu_t > \kappa t, \Psi_{S_s} \in (0, \rho), \overline{\tau}_B - \sigma > t, 0 \leq s \leq \mu_t\big) \\ &\quad \leq \widehat{\mathbf{P}}\big(\Psi_0 + H_s + 3rs + \widehat{P}_s \in (0, \rho), 0 \leq s \leq \kappa t\big) \\ &\quad = \widehat{\mathbf{P}}\big(\widehat{H}_s + \widehat{P}_s \in (0, \rho), 0 \leq s \leq \kappa t\big), \end{aligned} \tag{3.21}$$

where H is a standard Brownian motion (in particular, $H_s = M_{S_s}$ for $s < \mu_t$), $\widehat{P}_t$ is a process that satisfies $\widehat{P}_s - \widehat{P}_u \leq \varepsilon$, $u \leq s$, and $\widehat{H}_s = \Psi_0 + H_s + 3rs$. On the event indicated in the last line of (3.21), one has, for every integer $k < \kappa t$, that $\widehat{H}_k - \widehat{H}_{k-1} \geq -2$. Hence, the right-hand side of (3.21) can be estimated by $m_1 e^{-m_2 \kappa t}$, for some positive constants m_1 and m_2, independent of t and κ, ε, and as a result,

$$\widehat{\mathbf{P}}(F_t, \overline{\tau}_B - \sigma > t, \mu_t > \kappa t) \leq m_1 e^{-m_2 \kappa t}. \tag{3.22}$$

Next, on the event $F_t \cap \{\overline{\tau}_B - \sigma > t, \mu_t \leq \kappa t\}$ we have $\int_0^t m_s^2 (1 - \check{\alpha}_s - \check{\delta}_s)^2 \, ds \leq \kappa t$, thus

$$\int_0^t m_s^2 (1 - \check{\alpha}_s)^2 \, ds \leq 2\kappa t + 2 \int_0^t m_s^2 \check{\delta}_s^2 \, ds \leq (2\kappa + 16 \psi_{1,\infty}^2 \psi_{2,\infty}^2 \varepsilon^2) t \leq \frac{t}{4},$$

where we assumed without loss that $2\kappa + 16 \psi_{1,\infty}^2 \psi_{2,\infty}^2 \varepsilon^2 \leq 1/4$. Consequently, $\int_0^t m_s (1 - \check{\alpha}_s) \, ds \leq t/2$. Using $m_s \geq 1/2$, we have $\int_0^t (1 + \check{\alpha}_s) \, ds \geq 2t - t = t$, whence, letting c^0 be so large that $c^0 - 8r > 8r$,

$$3r\mu_t + \widetilde{P}_t \leq 3rt - 4rt = -rt,$$

where we used $\mu_t \leq \kappa t \leq t$. Using (3.16), on this event we have $0 \leq \Psi_t \leq \Psi_0 + M_t - rt + \widetilde{Q}_t$. Hence, recalling that $\widetilde{Q}_t \leq \varepsilon$ and $\Psi_0 \leq \psi_{1,\infty} \varepsilon$, denoting $m_0 =$

$\psi_{1,\infty}+1$, and letting γ be the (t-dependent) stopping time $\gamma=\inf\{s:\mu_s>\kappa t\}$, we have, for $t\ge r^{-1}m_0\varepsilon$,

$$\begin{aligned}\widehat{\mathbf{P}}(F_t,\overline{\tau}_B-\sigma>t,\mu_t\le\kappa t)&\le\widehat{\mathbf{P}}(M_t\ge rt-m_0\varepsilon,\mu_t\le\kappa t)\\&\le\widehat{\mathbf{P}}(M_{t\wedge\gamma}\ge rt-m_0\varepsilon)\\&\le\frac{m_3\widehat{\mathbf{E}}[\langle M\rangle^2_{t\wedge\gamma}]}{(rt-m_0\varepsilon)^4}\le\frac{m_3(\kappa t)^2}{(rt-m_0\varepsilon)^4}.\end{aligned}\tag{3.23}$$

In the second inequality above, we used the fact that $\mu_t\le\kappa t$ implies $\gamma\ge t$, and in the third we used Burkholder's inequality. In particular, m_3 does not depend on t or κ (which will allow us to use this estimate more efficiently in Step 3 below). Combining (3.20), (3.21) and (3.23) we obtain the statement in Step 1. $\square$

PROOF OF STEP 3. We begin by observing that, from (3.21),

$$\widehat{\mathbf{P}}(F_t,\overline{\tau}_B-\sigma>t,\mu_t>\kappa t)\le\mathbf{P}(\widehat{H}_s+\widehat{P}_s>0,0\le s\le\kappa t),\tag{3.24}$$

and since $\Psi_0+\widehat{P}_s\le m_0\varepsilon$, this probability is bounded by

$$p(\varepsilon,\kappa t):=\mathbf{P}(m_0\varepsilon+H_s+3rs>0,0\le s\le\kappa t).$$

The latter converges to zero as $\varepsilon\to0$ (for fixed κ and t). Let $\overline{\vartheta}$ be a modulus such that $p(\varepsilon,\overline{\vartheta}(\varepsilon))\le\overline{\vartheta}(\varepsilon)$, and $\frac{1}{2}\overline{\vartheta}(\varepsilon)^{1/4}\ge m_0\varepsilon$. Taking $t=r^{-1}(\overline{\vartheta}(\varepsilon))^{1/4}$ and $\kappa=r\overline{\vartheta}(\varepsilon)^{3/4}$, combining (3.23) and (3.24),

$$\widehat{\mathbf{P}}\big(F_t,\overline{\tau}_B-\sigma>r^{-1}\overline{\vartheta}(\varepsilon)^{1/4}\big)\le\overline{\vartheta}(\varepsilon)+\frac{m_3\overline{\vartheta}(\varepsilon)^2}{(1/2\overline{\vartheta}(\varepsilon)^{1/4})^4}=(1+16m_3)\overline{\vartheta}(\varepsilon).$$

Using the above estimate in (3.20), Step 3 follows. $\square$

PROOF OF STEP 4. For $a>0$, let τ_a and τ_0 denote the first time $[a,\infty)$, and, respectively, $(-\infty,0]$, is hit by $\widehat{H}$. Since $\widehat{H}$ is a Brownian motion (with drift $3r$) starting from $\widehat{H}(0)\le\psi_{1,\infty}\varepsilon$, we have that $\mathbf{P}(\tau_{\rho-\varepsilon}\le\tau_0)$ converges to zero as $\varepsilon\to0$. The proof is completed on noting that

$$\widehat{\mathbf{P}}(\overline{\tau}_\rho\le\overline{\tau}_G)\le\mathbf{P}(\tau_{\rho-\varepsilon}\le\tau_0),$$

which follows from (3.16), (3.17), the relation $\widehat{H}_s=\Psi_0+M_{S_s}+3r\mu_{S_s}$ for all $s<\mu_\infty\equiv\sup_{t\ge0}\mu_t$ and observing that on the set where $\sigma_0=\sup\{S_s:s<\mu_\infty\}<\infty$ we have that $M_t+3r\mu_t=M_{\sigma_0}+3r\mu_{\sigma_0}$, for $t\ge\sigma_0$. $\square$

4. Analysis of the game with bounded controls. The main result of this section, Theorem 4.1, implies Lemma 2.3. Fix k,l such that $\min\{k,l\}\ge\max\{c^0,d^0,k_{n_2},l_{n_2}\}$, where n_2 is as in Theorem 3.1. Throughout this section, (k,l) will be omitted from the notation. As in the previous section, only simple controls and strategies will be used. Recall that

$$\Phi(a,b,c,d;p,S)=-\tfrac{1}{2}(a-b)'S(a-b)-(c+d)(a+b)\cdot p.$$

Fix $\gamma \in [0, 1)$ and write

$$\Lambda_\gamma^+(p, S) = \max_{|a|=1, 0\le c\le k} \min_{|b|=1, 0\le d\le l} \Phi(a, b, c, d; p, S) - \frac{\gamma^2}{2}\operatorname{Tr}(S),$$

$$\Lambda_\gamma^-(p, S) = \min_{|b|=1, 0\le d\le l} \max_{|a|=1, 0\le c\le k} \Phi(a, b, c, d; p, S) - \frac{\gamma^2}{2}\operatorname{Tr}(S),$$

and consider the equations

$$\begin{cases} \Lambda_\gamma^+(Du, D^2u) - h = 0, & \text{in } G, \\ u = g, & \text{on } \partial G, \end{cases} \tag{4.1}$$

$$\begin{cases} \Lambda_\gamma^-(Du, D^2u) - h = 0, & \text{in } G, \\ u = g & \text{on } \partial G. \end{cases} \tag{4.2}$$

We will write V^γ and, respectively, U^γ for the functions V_{kl}^γ and U_{kl}^γ introduced at the beginning of Section 3.

THEOREM 4.1. *For each* $\gamma \in [0, 1)$, *one has the following*:

(i) *The function* U^γ *uniquely solves* (4.1).
(ii) *The function* V^γ *uniquely solves* (4.2).

The proof of the theorem is based on a result on a finite time horizon, Proposition 4.1, in which we adopt a technique of [12]. Given a function $u \in \mathcal{C}(\overline{G})$, $x_0 \in \overline{G}$, $T \ge 0$, and $Y \in M^0$, $Z \in M^0$, let

$$J^\gamma(x_0, T, u, Y, Z) = \mathbf{E}\biggl[\int_0^{T\wedge\tau^\gamma} h(X_s^\gamma)\,ds + u(X_{T\wedge\tau^\gamma}^\gamma)\biggr], \tag{4.3}$$

where X^γ and $\tau^\gamma = \tau(x_0, Y, Z)$ are as introduced in Section 3 with $X_0 = x_0$, $Y = (A, C)$ and $Z = (B, D)$.

PROPOSITION 4.1. *Let* $x_0 \in \overline{G}$, $T \in [0, \infty)$ *and* $\gamma \in [0, 1)$. *Let* $u \in \mathcal{C}(\overline{G})$.

(i) *If u is a subsolution of* (4.1), *then*

$$u(x_0) \le \sup_{\alpha\in\Gamma_{lk}^0} \inf_{Z\in M_l^0} J^\gamma(x_0, T, u, \alpha[Z], Z). \tag{4.4}$$

(ii) *If u is a supersolution of* (4.1), *then*

$$u(x_0) \ge \sup_{\alpha\in\Gamma_{lk}^0} \inf_{Z\in M_l^0} J^\gamma(x_0, T, u, \alpha[Z], Z). \tag{4.5}$$

(iii) *If u is a subsolution of* (4.2), *then*

$$u(x_0) \le \inf_{\beta\in\Gamma_{kl}^0} \sup_{Y\in M_k^0} J^\gamma(x_0, Y, T, u, \beta[Y]). \tag{4.6}$$

(iv) *If u is a supersolution of* (4.2), *then*

$$u(x_0) \geq \inf_{\beta \in \Gamma^0_{kl}} \sup_{Y \in M^0_k} J^\gamma(x_0, T, u, Y, \beta[Y]). \tag{4.7}$$

Before proving Proposition 4.1, we show how it implies the theorem.

PROOF OF THEOREM 4.1. We only prove (i) since the proof of (ii) is similar. We first argue that any solution of (4.1) must equal U^γ, and then show that a solution exists. Let a solution u of (4.1) be given. Fix $x_0 \in \overline{G}$ and $\varepsilon > 0$. Fix $\alpha \in \Gamma^0_{lk}$ such that

$$U^\gamma(x_0) \leq \inf_{Z \in M^0_l} J^{x_0}_\gamma(\alpha, Z) + \varepsilon. \tag{4.8}$$

By Proposition 4.1(ii),

$$u(x_0) \geq \inf_{Z \in M^0_l} j^\gamma(T, Z), \tag{4.9}$$

where we denote

$$j^\gamma(T, Z) = J^\gamma(x_0, T, u, \alpha[Z], Z).$$

For the rest of the proof, we suppress γ from the notation. Lemma 3.1 shows that there is $m_1 < \infty$ such that, for every $T \in [0, \infty)$, $\inf_{Z \in M^0_l} j(T, Z) \leq m_1$. Letting $M(T) = \{Z \in M^0_l : j(T, Z) \leq c_1\}$, it follows from the lower bound on h that for some $T < \infty$ that does not depend on Z, one has $\mathbf{P}(\tau > T) < \varepsilon$ for all $Z \in M(T)$, where $\tau = \tau^{x_0}(\alpha, Z)$. Fix such a T. Given $Z \in M(T)$, let $\widehat{Z} \in M^0_l$ be equal to Z on $[0, T)$, and let it assume the constant value $(a^0, 1)$ on $[T, \infty)$. Clearly $j(T, \widehat{Z}) = j(T, Z)$. Also, by Lemma 3.1, denoting $\widehat{\tau} = \tau^{x_0}(\alpha, \widehat{Z})$, we have

$$\mathbf{E}[(\widehat{\tau} - T)^+ | \widehat{\tau} > T] \leq m_2$$

for some constant m_2 independent of ε and T. By (4.3), (4.8), the definition of the payoff, and using the boundary condition $u|_{\partial G} = g$, we have for some $m_3 \in (0, \infty)$

$$\begin{aligned} U(x_0) &\leq J^{x_0}(\alpha, \widehat{Z}) + \varepsilon \\ &\leq J(x_0, T, u, \alpha[\widehat{Z}], \widehat{Z}) + m_3\{\mathbf{E}[(\widehat{\tau} - T)^+] + \mathbf{P}(\widehat{\tau} > T)\} + \varepsilon. \end{aligned}$$

Using $\mathbf{P}(\widehat{\tau} > T) = \mathbf{P}(\tau > T)$ yields $U(x_0) \leq j(T, \widehat{Z}) + m_4\varepsilon = j(T, Z) + m_4\varepsilon$. Note that the infimum of $j(T, Z)$ over M^0_l is equal to that over $M(T)$. Thus, using (4.9) and sending $\varepsilon \to 0$ proves that $U(x_0) \leq u(x_0)$.

To obtain the reverse inequality, fix $x_0 \in \overline{G}$. From Lemma 3.1, there exists $m_5 < \infty$ and $Z_1 \in M^0_l$ such that, for every α,

$$J^{x_0}(\alpha, Z_1) \leq m_5.$$

Denote $N(\alpha) = \{Z : J^{x_0}(\alpha, Z) \leq m_5\}$. Clearly, for each α, the infimum of $J^{x_0}(\alpha, Z)$ over all $Z \in M_l^0$ is equal to that over $Z \in N(\alpha)$. Hence,

$$U(x_0) \geq \inf_{Z \in N(\alpha)} J^{x_0}(\alpha, Z), \qquad \alpha \in \Gamma_{lk}^0.$$

Using the positive lower bound on h as before, it follows that there exists a function $r : [0, \infty) \to [0, \infty)$ with $\lim_{T \to \infty} r(T) = 0$, such that for every α and $Z \in N(\alpha)$ we have $\mathbf{P}(\tau^{x_0}(\alpha, Z) > T) \leq r(T)$. Therefore, for some $m_6 \in (0, \infty)$

$$J^{x_0}(\alpha, Z) \geq J(x_0, T, u, \alpha[Z], Z) - m_6 r(T), \qquad \alpha \in \Gamma_{lk}^0, Z \in N(\alpha).$$

In conjunction with Proposition 4.1(i), this shows that $U(x_0) \geq u(x_0) - m_6 r(T)$. Since T is arbitrary, we obtain $U(x_0) \geq u(x_0)$.

Finally, we argue existence of solutions to (4.1). Let us write $(4.1)_\gamma$ for (4.1) with a specific γ. For $\gamma \in (0, 1)$, existence of solutions to $(4.1)_\gamma$ follows from Theorem 1.1 of [5]. To handle the case $\gamma = 0$, we will use the fact that any uniform limit, as $\gamma \to 0$, of solutions to $(4.1)_\gamma$ is a solution to $(4.1)_0$. This fact follows by a standard argument, that we omit. Now, since for $\gamma \in (0, 1)$ we have existence, the uniqueness statement established above shows that U^γ solves $(4.1)_\gamma$. From Theorem 3.1, we have that the family $\{U^\gamma, \gamma \in (0, 1)\}$ is equicontinuous, and thus a uniform limit of solutions, and in turn a solution to $(4.1)_0$, exists. $\square$

In the rest of this section, we prove Proposition 4.1.

Let G_n be a sequence of domains compactly contained in G and increasing to G. Let J_n^γ be defined as J^γ of (4.3), with $\tau^\gamma = \tau^\gamma(x_0, Y, Z)$ replaced by $\tau_n^\gamma = \tau_n^\gamma(x_0, Y, Z)$, where

$$\tau_n^\gamma = \inf\{t : X_t^\gamma \in \partial G_n\}.$$

LEMMA 4.1. *For every n, and $\gamma \in [0, 1)$ Proposition 4.1 holds with J^γ replaced by J_n^γ.*

PROOF. We follow the proof of [12], Lemma 2.3 and Theorem 2.1. Assume without loss that $G_0 \subset\subset G_1 \subset\subset G_2 \subset\subset G$. We will prove the lemma for $n = 0$. Since the claim is trivial, if $x_0 \notin G_0$, assume $x_0 \in G_0$. In this proof only, write τ for τ_0^γ, the exit time of X^γ from G_0. Fix $\widetilde{\gamma} > \gamma$, let $\widetilde{\tau} = \tau_1^{\widetilde{\gamma}}$ and $\sigma = \tau \wedge \widetilde{\tau}$. For $\varepsilon > 0$, consider the sup convolution

$$u_\varepsilon(x) = \sup_{\xi \in \mathbb{R}^m} \left\{ u(\xi) - \frac{|\xi - x|^2}{2\varepsilon} \right\}, \qquad x \in G_2,$$

where, in the above equation only, u is extended to $\mathbb{R}^m$ by setting $u = 0$ outside G. It is easy to see that there exists ε_0 such that the supremum is attained inside G for all $(x, \varepsilon) \in G_2 \times (0, \varepsilon_0)$. The standard mollification $u_\varepsilon^\delta : \overline{G}_1 \to \mathbb{R}$ of $u_\varepsilon : G_2 \to \mathbb{R}$ is well defined, provided that δ is sufficiently small. The result [12], Lemma 2.3, for the smooth function u_ε^δ and the argument in the proof of [12], Theorem 2.1, show

$$u_\varepsilon^\delta(x_0) \le \sup_{\alpha\in\Gamma^0} \inf_{Z\in M^0} \mathbf{E}\biggl[\int_0^{T\wedge\sigma} h(X_s^{\widetilde{\gamma}})\,ds + u_\varepsilon^\delta(X_{T\wedge\sigma}^{\widetilde{\gamma}})\biggr] + \rho(\varepsilon,\delta,\gamma,\widetilde{\gamma}),$$

where $\lim_{\varepsilon\to 0}\lim_{\widetilde{\gamma}\to\gamma}\lim_{\delta\to 0}\rho(\varepsilon,\delta,\gamma,\widetilde{\gamma}) = 0$. We remark here that Lemma 2.3 of [12] is written for the case where u is a subsolution of a PDE of the form (4.1) on all of $\mathbb{R}^m$ and $T\wedge\sigma$ is replaced by T, however the proof with u and $T\wedge\sigma$ as in the current setting can be carried out in exactly the same way. Since G_0 is compactly contained in G_1, we have that for every $\theta > 0$

$$\sup_{\alpha\in\Gamma^0}\sup_{Z\in M^0}\Bigl\{\mathbf{P}\Bigl(|T\wedge\sigma - T\wedge\tau| + \sup_{0\le s\le T}|X_s^{\widetilde{\gamma}} - X_s^{\gamma}| > \theta\Bigr)\Bigr\}$$

converges to 0 as $\widetilde{\gamma}\to\gamma$. Moreover, $u_\varepsilon^\delta \to u_\varepsilon$ as $\delta\to 0$ and $u_\varepsilon \to u$ as $\varepsilon\to 0$, where in both cases, the convergence is uniform on $\overline{G}_0$ (see ibid.). Hence, the result follows on taking $\delta\to 0$, then $\widetilde{\gamma}\to\gamma$ and finally $\varepsilon\to 0$. $\square$

PROOF OF PROPOSITION 4.1. The main argument is similar to that of Theorem 2.1, and so we omit some of the details. We will prove only item (iv) of the proposition, since the other items can be proved in a similar way.

Fix x and T. Let u be a supersolution of (4.2). Let n be large enough so that $\mathrm{dist}(\partial G_n, \partial G) < \rho_1$. Write $j_n^\gamma(Y,\beta)$ for $J_n^\gamma(x,T,u,Y,\beta[Y])$ and $j^\gamma(Y,\beta)$ for $J^\gamma(x,T,u,Y,\beta[Y])$. Below we will keep γ in the notation only if there is scope for confusion. By Lemma 4.1, $u(x)\ge v_n := \inf_\beta \sup_Y j_n(Y,\beta)$, for every n. We need to show $u(x)\ge v := \inf_\beta\sup_Y j(Y,\beta)$.

Fix $\varepsilon > 0$. Let β_n be such that

$$\sup_Y j_n(Y,\beta_n) \le v_n + \varepsilon \tag{4.10}$$

and let $\tau_1^n(Y) = \tau_n^\gamma(x,Y,\beta_n[Y])$, $Y\in M_l^0$. Let $\widetilde{\beta}_n$ be constructed from β_n as in the proof of Theorem 2.1, where in particular, $\beta_n[Y]$ and $\widetilde{\beta}_n[Y]$ differ only on $[\tau_1^n,\infty)$, by which $j_n(Y,\widetilde{\beta}_n) = j_n(Y,\beta_n)$. Choose Y_n such that

$$v \le \sup_Y j(Y,\widetilde{\beta}_n) \le j(Y_n,\widetilde{\beta}_n) + \varepsilon$$

and set $\tau_2^n(Y) = \tau^\gamma(x,Y,\widetilde{\beta}_n[Y])$. Then

$$v - v_n - 2\varepsilon \le j(Y_n,\widetilde{\beta}_n) - j_n(Y_n,\widetilde{\beta}_n) =: \delta_n.$$

Denote $X_n = X^x(Y_n,\widetilde{\beta}_n)$. Using Lemma 3.2,

$$0\le \tau_2^n - \tau_1^n < \varepsilon \quad\text{and}\quad |X_n(\tau_1^n\wedge T) - X_n(\tau_2^n\wedge T)| < \varepsilon$$

with probability tending to 1 as $n\to\infty$. It now follows from the definition of J_n and J [cf. (4.3)] that $\limsup_n \delta_n \le \rho(\varepsilon)$ for some modulus ρ. Since ε is arbitrary, this proves the result. $\square$

5. Concluding remarks.

5.1. *Identity* (1.4). Recall from (2.1) that

$$\Phi(a,b,c,d;p,S) = -\tfrac{1}{2}(a-b)'S(a-b) - (c+d)(a+b)\cdot p \tag{5.1}$$

and denote

$$\Lambda^+(p,S) = \sup_{|b|=1, 0\le d<\infty} \inf_{|a|=1, 0\le c<\infty} \Phi(a,b,c,d;p,S), \tag{5.2}$$

$$\Lambda^-(p,S) = \inf_{|a|=1, 0\le c<\infty} \sup_{|b|=1, 0\le d<\infty} \Phi(a,b,c,d;p,S) \tag{5.3}$$

[compare with (2.2) and (2.3)]. The following proposition establishes identity (1.4) that, as discussed in the introduction, allows one to view the infinity-Laplacian equation as a Bellman–Issacs type equation. The result states that for the SDG of Section 1.2, the associated Isaacs condition, $\Lambda^+ = \Lambda^-$, holds. Although we do not make use of it in our proofs, such a condition is often invoked in showing that the game has value (cf. [7, 12]).

PROPOSITION 5.1. *For $p \in \mathbb{R}^m$, $p \neq 0$ and $S \in \mathcal{S}(m)$, $\Lambda^+(p,S) = \Lambda(p,S)$ and $\Lambda^-(p,S) = \Lambda(p,S)$. In particular, identity* (1.4) *holds.*

PROOF. We will only show $\Lambda^- = \Lambda$ (the proof of $\Lambda^+ = \Lambda$ being similar). Fix p, S, and omit them from the notation. Write $\mathcal{H}_k$ for $\{(a,c) \in \mathcal{H} : c \le k\}$ and $\phi(y,z)$ for $\Phi(a,b,c,d)$, where $y = (a,c)$, $z = (b,d)$. Given $\delta > 0$ let k be such that $\Lambda^- \ge \inf_{y\in\mathcal{H}_k} \sup_{z\in\mathcal{H}} \phi(y,z) - \delta$. Then

$$\Lambda^- \ge \inf_{y\in\mathcal{H}_k} \sup_{z\in\mathcal{H}_l} \phi(y,z) - \delta = \Lambda^-_{kl} - \delta.$$

Thus, by Lemma 2.1, $\Lambda^- \ge \Lambda$.

Next, let $\overline{\phi}(y) = \sup_{z\in\mathcal{H}} \phi(y,z)$. Fix $\delta \in (0,\infty)$, let $y_\delta = (\overline{p}, \delta^{-1})$, where $\overline{p} = p/|p|$, and let $z_\delta = (b_\delta, d_\delta) \in \mathcal{H}$ be such that $\overline{\phi}(y_\delta) \le \phi(y_\delta, z_\delta) + \delta$. Then

$$\begin{aligned}\Lambda^- \le \overline{\phi}(y_\delta) &\le -\tfrac{1}{2}(\overline{p} - b_\delta)'S(\overline{p} - b_\delta) - (\delta^{-1} + d_\delta)(\overline{p} + b_\delta)\cdot p + \delta \\ &\le -\tfrac{1}{2}(\overline{p} - b_\delta)'S(\overline{p} - b_\delta) + \delta.\end{aligned}$$

Note that b_δ must converge to $-\overline{p}$ or else the middle inequality above will say $\Lambda^- = -\infty$, contradicting the bound $\Lambda^- \ge \Lambda$. Letting $\delta \to 0$, we now have from the third inequality that $\Lambda^- \le \Lambda$. The result follows. $\square$

5.2. *Limit trajectory under a nearly optimal play.* In [10], the authors raise questions about the form of the limit trajectory under optimal play of the Tug-of-

War game, as the step size approaches zero (see Section 7 therein). It is natural to ask, similarly, whether one can characterize (near) optimal trajectories for the SDG studied in the current paper. Let V be as given in (1.10). Let $x \in \overline{G}$ and $\delta > 0$ be given. We say that a policy $\beta \in \Gamma$ is δ-optimal for the lower game and initial condition x if $\sup_{Y \in M} J^x(Y, \beta) \le V(x) + \delta$. When a strategy $\beta \in \Gamma$ is given, we say that a control $Y \in M$ is δ-optimal for play against β with initial condition x, if $J^x(Y, \beta) \ge \sup_{Y' \in M} J^x(Y', \beta) - \delta$. A pair (Y, β) is said to be a δ-optimal play for the lower game with initial condition x, if β is δ-optimal for the lower game and Y is δ-optimal for play against β (both considered with initial condition x). One may ask whether the law of the process X^δ, under an arbitrary δ-optimal play (β^δ, Y^δ), converges to a limit law as $\delta \to 0$; whether this limit law is the same for any choice of such (β^δ, Y^δ) pairs; and finally, whether an explicit characterization of this limit law can be provided. A somewhat less ambitious goal, that is the subject of a forthcoming work [3] is the characterization of the limit law of X^δ under *some* choice of a δ-optimal play. The result from [3] states the following.

THEOREM 5.1. *Suppose that V is a $C^2(\overline{G})$ function and $DV \neq 0$ on $\overline{G}$. Assume there exist uniformly continuous bounded extensions, p and q of $\frac{Du}{|Du|}$ and $\frac{1}{|Du|^2}(D^2 u Du - \Delta_\infty u Du)$, respectively, to $\mathbb{R}^m$ such that, for every $x \in \mathbb{R}^m$, weak uniqueness holds for the SDE*

$$dX_t = 2p(X_t)\,dW_t + 2q(X_t)\,dt, \qquad X_0 = x.$$

Fix $x \in \overline{G}$ and let X and τ denote such a solution and, respectively, the corresponding exit time from G. Then, given any sequence $\{\delta_n\}_{n \ge 1}$, $\delta_n \downarrow 0$, there exists a sequence of strategy-control pairs $(\beta^n, Y^n) \in M \times \Gamma$, $n \ge 1$, with the following properties:

(i) *For every n, the pair (β^n, Y^n) forms a δ_n-optimal play for the lower game with initial condition x.*

(ii) *Denoting $X^n = X(x, Y^n, \beta^n)$ and $\tau^n = \tau(x, Y^n, \beta^n)$, one has that $(X^n(\cdot \wedge \tau^n), \tau^n)$ converges in distribution to $(X(\cdot \wedge \tau), \tau)$, as a sequence of random variables with values in $C([0, \infty) : \overline{G}) \times [0, \infty]$.*

An analogous result holds for the upper game.

A sufficient condition for the uniqueness to hold is that $D^2 u$ is Lipschitz on $\overline{G}$, since then both p and q are Lipschitz, and thus admit bounded Lipschitz extensions to $\mathbb{R}^m$.

REFERENCES

[1] ARONSSON, G. (1967). Extension of functions satisfying Lipschitz conditions. *Ark. Mat.* **6** 551–561. MR0217665

[2] ARONSSON, G. (1972). A mathematical model in sand mechanics: Presentation and analysis. *SIAM J. Appl. Math.* **22** 437–458. MR0308527

[3] ATAR, R. and BUDHIRAJA, A. (2008). On near optimal trajectories for a game associated with the ∞-Laplacian. *Trans. Amer. Math. Soc.* **360** 77–101.

[4] BARRON, E. N., EVANS, L. C. and JENSEN, R. (2009). The infinity Laplacian, Aronsson's equation and their generalizations. Preprint.

[5] CRANDALL, M. G., KOCAN, M., LIONS, P. L. and SWIECH, A. (1999). Existence results for boundary problems for uniformly elliptic and parabolic fully nonlinear equations. *Electron. J. Differential Equations* 24. MR1696765

[6] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence*. Wiley, New York. MR838085

[7] FLEMING, W. H. and SOUGANIDIS, P. E. (1989). On the existence of value functions of two-player, zero-sum stochastic differential games. *Indiana Univ. Math. J.* **38** 293–314. MR997385

[8] JENSEN, R. (1993). Uniqueness of Lipschitz extensions: Minimizing the sup norm of the gradient. *Arch. Ration. Mech. Anal.* **123** 51–74. MR1218686

[9] KOHN, R. V. and SERFATY, S. (2006). A deterministic-control-based approach to motion by curvature. *Comm. Pure Appl. Math.* **59** 344–407. MR2200259

[10] PERES, Y., SCHRAMM, O., SHEFFIELD, S. and WILSON, D. B. (2009). Tug-of-war and the infinity Laplacian. *J. Amer. Math. Soc.* **22** 167–210. MR2449057

[11] SONER, H. M. and TOUZI, N. (2003). A stochastic representation for mean curvature type geometric flows. *Ann. Probab.* **31** 1145–1165. MR1988466

[12] SWIECH, A. (1996). Another approach to the existence of value functions of stochastic differential games. *J. Math. Anal. Appl.* **204** 884–897. MR1422779

DEPARTMENT OF ELECTRICAL ENGINEERING
TECHNION—ISRAEL INSTITUTE OF TECHNOLOGY
HAIFA 32000
ISRAEL
E-MAIL: atar@ee.technion.ac.il

DEPARTMENT OF STATISTICS
AND OPERATIONS RESEARCH
UNIVERSITY OF NORTH CAROLINA
CHAPEL HILL, NORTH CAROLINA 27599
USA
E-MAIL: budhiraj@email.unc.edu