Annals of Nuclear Energy 111 (2018) 608–615

Contents lists available at ScienceDirect

Annals of Nuclear Energy

journal homepage: www.elsevier.com/locate/anucene

Modeling neutron count distribution in a subcritical core by stochastic differential equations

Chen Dubi [a,*], Rami Atar [b]

[a] Physics Department, Nuclear Research Center of the Negev, P.O.B. 9001, Beer Sheva, Israel
[b] Viterbi Faculty of Electrical Engineering, Technion–Israel Institute of Technology, Haifa 32000, Israel

ARTICLE INFO

Article history:
Received 8 December 2016
Received in revised form 12 September 2017
Accepted 21 September 2017

Keywords:
Point reactor kinetics
Diffusion approximations
Stochastic differential equations

ABSTRACT

Reactor noise, caused both by the probabilistic nature of the fission chains and external reactivity noises, is one of the basic topics in nuclear science and engineering, both in theory and practice. Classical approaches to modeling this noise and neutron count distribution in the detection system rely on the stochastic transport equation for the probability generating function and on transfer function response to random perturbations. In recent years, a third modeling approach has been proposed, relying on *Ito stochastic differential equations*, which enjoys the tractability that the first aforementioned approach has, and at the same time accounts for fluctuations, by modeling noise in terms of Brownian motion. This paper develops the latter approach to incorporate the stochasticity in the detection process to the model equations. The resulting neutron count distributions are explicitly computable.

As an application of our approach we present a straightforward derivation of the well-known Feynman-Y formula. We then propose an alternative to the traditional sampling scheme of this formula, based on *mean absolute deviation*, known from the statistics literature to be more robust than the *mean square deviation* estimator. The study focuses on a single energy point model and neglects the effect of the delayed neutrons. Extensions of the approach to multiple energy levels and the incorporation of delayed neutrons are discussed, as well as further applications of the approach and its advantages over existing diffusion scale approximations.

1. Introduction

Reactor noise and neutron flux fluctuation is one of the basic topics in nuclear science and engineering, both in theory and practice. Applications of the theory of neutron fluctuation may be found both in monitoring and measurements (Uhrig, 1970), and in non destructive assay of special nuclear materials (Ensslin et al., 1998). Fluctuations in the neutron population size may be attributed to two types of statistical noises: Internal noises, governed by the statistical nature of the neutron interactions, and external noises, reflecting stochasticity of other elements of the system, such as temperature fluctuations, mechanical instabilities, electronic noise in the monitoring system and more (Williams, 1974).

Reactor noise and *neutron fluctuations* are general terms used to describe the modeling and sampling of higher moments of the neutron population distribution in multiplying systems due to both internal and external factors. The theory of reactor noise is very different between the so called *Zero Power Reactors* (ZPR), where the main contribution to the statistical variance is due to the internal factors, and full scale power reactors, where the statistical variance is largely dominated by the external factors. Due to the many differences between the internal and external noises, the two are traditionally modeled differently and analyzed using distinct mathematical tools. Internal noises are studied through the probability generating function and the Kolmogorov equation (a comprehensive overview may be found in Pazsit and Pal (2008)), when external noises are often treated as random perturbation on the point reactor kinetics equation (PKE), and are typically analyzed through transfer function input/output analysis (Williams, 1974). In particular, whereas the perturbation is considered random, the analysis does not consider the noise as a true stochastic process, and the results are typically specified in terms of the output response to a fixed perturbation. Indeed, the transfer function is very useful in determining the amplitude of the response to a random noise but says little on the probabilistic characteristics of the response.

* Corresponding author.
E-mail address: chendb331@gmail.com (C. Dubi).

https://doi.org/10.1016/j.anucene.2017.09.040
0306-4549/

Hayes and Allen (2005) proposed a new modeling approach for reactor noise based on the diffusion approximations,[1] by which the *central limit theorem* (CLT) provides a model for the stochastic fluctuations. The model is expressed in terms of *Ito stochastic differential equations* (SDE) (Karatzas and Shreve, 1991). In a sense, this modeling scheme is intermediate, lying between the deterministic point reactor kinetic equation and the full stochastic transport equation. Formally it corresponds to versions of the point reactor kinetic equation with terms accounting for stochastic perturbations.[2]

The goal of this paper is to develop the SDE approach by incorporating stochasticity associated with the detection process. Despite the crucial role played by the stochastic detection process, existing SDE models have accounted only for the stochastic fluctuations in the neutron population. However, a model that lacks the detection aspect is arguably incomplete, since it is only possible to infer the population size through the detector response. Thus coupling the detector response with the equation for the population size appears to be of utmost importance for any practical implementation of the approach.

This paper offers three main contributions, that are all related to the detection stochasticity. (i) The incorporation of the detection process into the SDE model, in a way that accounts for its stochastic nature. The model is formulated as a *coupled pair* of SDE, cosisting of an equation for the neutron population and another for the detection count. (ii) Derivation of explicit formula for the Feynman variance-to-mean ratio (which also serves as a strong validation of the proposed model). (iii) Based on the proposed model, a discussion of an alternative method for sampling the Feynman-Y curve, via the mean average deviation.

The analysis in this paper is restricted to the single energy point model (an assumption that is also in force in Hayes and Allen (2005)), and the delayed neutrons are neglected (incorporating the delayed neutrons is doable, but extremely lengthy, and in the present context, with no real gain).

The paper is organized as follows. The remainder of the present section describes the motivation. In Section 2 some background on both reactor noise and stochastic analysis is given. Section 3, which constitutes the main contribution, introduces the SDE of relevance, and provides some basic analysis thereof. Section 4 is devoted to a proposed method of sampling the Feynman-Y function, that is theoretically based on our SDE model, for which we provide experimental evidence. The main idea of the method is to use the mean absolute deviation to estimate the variance, rather than the traditional mean squares. Section 5 lists our conclusions from this study.

1.1. Motivation

Introduced in the early 1950's in the seminal work of Feynman (1945) and having been covered by numerous textbooks on the subject since then, the modeling and analysis of reactor noise are important both in theory (Malinovitch and Dubi, 2015; Demeshko et al., 2016) and applications (Diniz and dos Sontas, 2006; Diniz and dos Sontas, 2002). Still, it is widely accepted that reactor noise is not fully understood. In his book from 1974, M. M.R. Williams states (Williams, 1974):

"*. . .noise analysis of power reactors is in its infancy due mainly to a lack of knowledge about the variety of noise mechanisms involved. . .In terms of the input - output concept we are not only ignorant of the nature of the input but in many cases of the system response function as well. . .the great number of noise sources and fluctuating parameters is the main stumbling-block for scientists in the field, especially for those who have become accustomed to the fascinating clearness of zero-power noise studies. At the same time it is precisely the difficulties that constitute the merit of the topic;...*"

Since 1974, the topic has been vastly studied by many contributors, but no fundamental breakthrough was achieved in our basic understanding of how a random fluctuation in the reactor parameters would propagate on to the power level and the neutron flux. Since the SDE model was introduced in Hayes and Allen (2005), the model was adopted by many contributors, including the following (to state a few): in Ha and Kim (2010), the model was extended to a stochastic PDE, allowing 1D spatial dependence of the neutron population, in Ha and Kim (2011), the reactor transient behavior was studied, in Allen (2013), the doubling time of a subcritical assembly was studied and in da Silva (2016), numeric solutions to the SDE were studied.

As already mentioned, in all previous work, the detection process was completely neglected. From the theoretical viewpoint of studying the population dynamics per se, this can be justified: the effect of the detection can merely be thought of as absorption. However, from a practical point of view, modeling the detection process is crucial, since the neutron detections are the only observable that can be directly linked to the neutron population.

2. Background

2.1. The point reactor kinetics equation

The Point reactor Kinetics Equation (PKE), describing the average neutron population, takes the form Ott and Neuhold (1985)

$$\frac{dN(t)}{dt} = \frac{\rho - \beta_{\text{eff}}}{\Lambda} N(t) + \sum_{j=1}^{\kappa} \lambda_j C_j(t) + S(t), \tag{1}$$

$$\frac{dC_j(t)}{dt} = -\lambda_j C_j(t) + \frac{\beta_j}{\Lambda} N(t), \tag{2}$$

where

N denotes the number of neutrons,

C_j the concentration of the jth delayed neutron group precursor,

ρ the reactivity,

Λ the generation time,

λ_j the decay constant of the jth delayed neutron group precursor,

β_j the fraction (in units of reactivity) of the jth delayed neutron group precursor,

β_{eff} the delayed neutron fraction (in units of reactivity), defined by $\beta_{\text{eff}} = \sum_{j=1}^{\kappa} \beta_j$.

This is one of the most basic equations in nuclear engineering, with numerous applications. In common practice we assume that there are $\kappa = 6$ delayed neutron groups, and the values of the parameters $\{\lambda_j\}_{j=1}^{6}, \{\beta_j\}_{j=1}^{6}$ may be found in the literature.

A simplified version of the PKE is the prompt reactivity model, where the delayed neutron fraction is treated as an external source, and the dynamics are governed by a simplified, point equation

$$\frac{dN(t)}{dt} = \frac{\rho - \beta_{\text{eff}}}{\Lambda} N(t) + S(t). \tag{3}$$

[1] It should be clarified that the use of the term *diffusion approximation* in this paper is different than its more standard use in the nuclear physics literature. That is, it refers to the identification of CLT-scale limits describing the neutron dynamics over time (as well as other stochastic processes), not to be confused with *spatial diffusion approximation*, associated with Fick's law, that is often used in relation to the Boltzmann equation.

[2] Diffusion scale approximations have been also used before in this field in the context of partial differential equations (PDE), specifically by appealing to the Fokker-Planck equation (Williams, 1974) (Ch. 5.6).

The term $\alpha = -(\rho - \beta_{\text{eff}})/\Lambda$ is often referred to as the *decay coefficient* (or sometimes the α *eigenvalue*) of the system.

Denoting by $\bar{\nu}$ the average number of neutrons emitted in a fission, the multiplication factor may be written as $k = p_f\bar{\nu}$, where p_f is the fission probability. Using the relation $\rho = (k-1)/k$, the coefficient of $N(t)$ in (3) is equal to $(p_f\bar{\nu} - 1)/\ell$, where ℓ is the average lifetime of a neutron. Transforming to the average lifetime scale by $t \to t/\ell$, we obtain

$$\frac{dN(t)}{dt} = (p_f\bar{\nu} - 1)N(t) + S(t). \tag{4}$$

In the above equation, the source term S was also normalized to the average life time scale. Eq. (4) may be derived using the reaction intensities. If we denote by λ_f the fission probability per time unit (or *fission intensity*), by λ_a the absorption intensity, by $\lambda = \lambda_a + \lambda_f$ the total intensity and by $S(t)$ external source intensity, then the neutron balance equation may be written as

$$dN(t) = -\lambda N(t)dt + \bar{\nu}\lambda_f N(t)dt + S(t)dt. \tag{5}$$

As stated, the PKE is a basic tool in analyzing the kinetics of research and power reactors. Yet, it only describes the mean field of the neutron population, without capturing its stochastic nature.

2.2. Stochastic differential equations

Dynamic evolution that involves randomness, viewed at diffusion scale, gives rise to SDE in a large variety of settings. This occurs in application fields such as population genetics, queuing networks, finance, communication systems, and theoretical physics. When the fluctuations of the dynamics are diffusive, working with SDE often simplifies their description considerably while keeping the essence. The method by which evolution dynamics are approximated by SDE is referred to in the literature as *diffusion approximation*. Diffusion approximations have been successfully applied in various application fields in all the areas alluded to above.

The two most basic processes underlying the neutron population dynamics are the Poisson process, that provides a natural model for the particle injection, absorption and detection, and branching processes, that model fission. Both processes lie in the classical realm of probability theory, and specifically, their scaling limits, including *law of large numbers* (LLN) and CLT, are well understood. It seems that (Pakes, 1971) was first to provide rigorous derivation of LLN and CLT results for the total progeny of subcritical branching processes with immigration. For nearly critical branching processes, the limiting behavior is given by continuous state branching process with immigration when the initial condition is getting large, a direction that started from Kawazu and Watanabe (1971). The state of the art is described in the recent book (Dawson and Li, 2012) (see eg., Theorem 3.43 there). In this work we are interested in the formulation and study of SDE that arise as diffusion approximations of nuclear dynamics.

An SDE is an equation of the form

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \qquad X_0 = x, \tag{6}$$

where the unknown is a stochastic process X, that has continuous sample paths taking values in $\mathbb{R}^d$, for some positive integer d; b and σ are given coefficients; and W is a d-dimensional Brownian motion (BM). A process X is regarded a solution if it satisfies, for every $t, X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s$, where the last term in this integral equation is an *Ito integral*. The special case where $\sigma = 0$ corresponds to an ordinary differential equation.

Questions of existence and uniqueness of solutions to SDE such as (6), their Markov structure, properties of the solutions, as well as solution methods have been of great interest and enjoyed a remarkable success ever since the 1960's, although the pioneering work goes back to Ito (1946, 1951) and Gihman (1947, 1950). The rich literature includes qualitative theory such as boundedness and stability of solutions, solution methods, representation of solutions, most notably via the Girsanov transformation, the study of fine properties of solutions, as well as the description of time evolution and steady state distribution by means of Kolmogorov's forward and backward equations; a small sample of books addressing the subject is Karatzas and Shreve (1991), Øksendal (2003), Gard (1988), Karlin and Taylor (1981) and Ethier and Kurtz (1986). in Hayes and Allen (2005), a refinement of Eq. (5), in the form of a SDE such as (6).

In a nutshell, the goal of this paper is to extend the model introduced in Hayes and Allen (2005) to a set of SDE's, incorporating the detection process and its inherent stochastic nature.

3. Modeling stochastic dynamics of neutron population by SDE

The main contribution of this paper is contained in this section, where we derive a set of SDE's for the neutron population dynamics coupled with the neutron detection.

under sub-critical branching. The model is constructed under three main assumptions. First, only internal noises are modeled. These consist of the stochastic fluctuations due to the Poissonian nature of the times of injections by the source, the randomness in the underlying branching mechanism, and the randomness associated with absorption. Second, we consider a single energy point model (an assumption that is also in the basis of the derivation of the RPK equation). Finally, we assume a prompt reactivity model, and thus neglect the effect of the delayed neutrons.

In Section 3.1, for sake of completeness, we partially repeat the the analysis in Hayes and Allen (2005) and provide a derivation of a basic diffusion approximation for the neutron population.[3] In Section 3.2, which forms the main contribution of our study, a finer model is derived, that accounts for the destruction associated with detection. The diffusion approximation in this case consists of a coupled pair of SDE. Section 3.3 substantiates the model just introduced by deriving from it the well-known Feynman-Y formula.

Notation related to random variables (RV) is as follows. For a RV X, we denote by $E(X)$ and $\text{Var}(X)$ its mean and variance, respectively. The covariance between two RVs X and Y will be denoted by $\text{Cov}(X, Y)$.

3.1. An SDE for the neutron population

Under the single energy point model, the neutron population is modeled in terms of three parameters:

1. The fission probability per time unit, denoted by λ_f.
2. The absorption probability per time unit, denoted by λ_a.
3. The distribution of the number of neutrons emitted in a fission (or the *neutron multiplicity*), denoted by $\{p(\nu)\}_{\nu=0}^{\nu_{max}}$. We will denote by $\bar{\nu}$ and $\overline{\nu^2}$ the first and, resp., second moments of this distribution.

Denote by $\lambda = \lambda_f + \lambda_a$ the reaction probability per time unit. This parameter can otherwise be characterized as the reciprocal average die-away time of a neutron. Moreover, $p_f = \lambda_f/\lambda$ and $p_a = \lambda_a/\lambda$ give the fission and absorption probabilities, respectively. Consider a short time interval $[t, t + \Delta t]$ and suppose one focuses on a specific neutron that is alive at time t. Let X denote the number of neutrons originating from that specific neutron, that are alive at time $t + \Delta t$. That is, let $X = 1$ on the event that the neutron has

[3] Notice that we neglect the effect of the delayed neutrons, while in Hayes and Allen (2005) it is not neglected.

made no reaction, let $X = 0$ on the event of absorption, and let X be the number of offspring on the event that fission has occurred. Let $Y = X - 1$ denote the increase in the number of neutrons of that particular neutron, by the end of the aforementioned time interval. Accordingly,

$$X = \begin{cases} 0 & \text{w.p. } \lambda p_a \Delta t, \\ 1 & \text{w.p. } 1 - \lambda \Delta t, \\ \nu & \text{w.p. } \lambda p_f p(\nu)\Delta t, \ \nu \geq 1. \end{cases}$$

This can otherwise be written as

$$X = \begin{cases} 0 & \text{w.p. } \lambda_a \Delta t + \lambda_f p(0)\Delta t, \\ 1 & \text{w.p. } 1 - \lambda \Delta t + \lambda_f p(1)\Delta t, \\ \nu & \text{w.p. } \lambda_f p(\nu)\Delta t, \ \nu \geq 2. \end{cases}$$

We can therefore compute

$$E[X] = 1 - \lambda \Delta t + \lambda_f p(q)\Delta t + \sum_{\nu \geq 2} \lambda_f p(\nu)\nu \Delta t = 1 - \lambda \Delta t + \lambda_f \overline{\nu}\Delta t,$$
$$E[X^2] = 1 - \lambda \Delta t + \lambda_f p(1)\Delta t + \sum_{\nu \geq 2} \lambda_f p(\nu)\nu^2 \Delta t = 1 - \lambda \Delta t + \lambda_f \overline{\nu^2}\Delta t.$$

Hence

$$E[Y] = E[X] - 1 = -\lambda \Delta t + \lambda_f \overline{\nu}\Delta t,$$

and

$$\begin{aligned} \mathrm{Var}(Y) = \mathrm{Var}(X) &= E[X^2] - E[X]^2 \\ &= -\lambda \Delta t + \lambda_f \overline{\nu^2}\Delta t + 2(\lambda - \lambda_f \overline{\nu})\Delta t \\ &= [\lambda + \lambda_f(\overline{\nu^2} - 2\overline{\nu})]\Delta t, \end{aligned}$$

where in the last display we have neglected terms of order $(\Delta t)^2$. With the notation

$$\alpha = \lambda - \lambda_f \overline{\nu}, \quad \tilde{\sigma} = \sqrt{\lambda + \lambda_f(\overline{\nu^2} - 2\overline{\nu})}, \tag{7}$$

we have

$$E[Y] = -\alpha \Delta t, \quad \mathrm{Var}(Y) = \tilde{\sigma}^2 \Delta t.$$

Now, recall that we denote by N_t the size of the neutron population at time t. Then one can associate a RV Y_i to the ith neutron, for $i = 1, 2, \ldots, N_t$, where $\{Y_i\}_{i \in \mathbb{N}}$ is a given independent, identically distributed sequence with Y_i equal in distribution to Y, for each i. Thus denoting the increase in the population size by $\Delta N_t = N_{t+\Delta t} - N_t$, the size of the population at time $t + \Delta t$ is given by

$$N_{t+\Delta t} = N_t + \Delta N_t = N_t + \sum_{i=1}^{N_t} Y_i.$$

We can analyze the last term above by appealing to the CLT, which states that, as $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{Y_i + \alpha \Delta t}{\tilde{\sigma}\sqrt{\Delta t}} \Rightarrow \mathcal{N}(0, 1),$$

where '$\Rightarrow$' denotes convergence in distribution. According to this, if n is a large number then one may approximate

$$\sum_{i=1}^{n} Y_i \stackrel{d}{=} -n\alpha \Delta t + \mathcal{N}(0, n\tilde{\sigma}^2 \Delta t),$$

where $\stackrel{d}{=}$ denotes equality in distribution. Since the number of neutrons is typically very large – starting from 10^9 in small ZPR up to 10^{19} in full scale power reactors – it is reasonable to use the above display with $n = N_t$. This gives the approximation

$$\Delta N_t = \sum_{i=1}^{N_t} Y_i \stackrel{d}{=} -\alpha N_t \Delta t + \mathcal{N}(0, N_t \tilde{\sigma}^2 \Delta t). \tag{8}$$

Next, if $\{W_t\}$ is a standard BM and ΔW_t denotes $W_{t+\Delta t} - W_t$, then we can use the fact that ΔW_t is distributed according to $\mathcal{N}(0, \Delta t)$ to write the above equality in distribution as

$$\Delta N_t \stackrel{d}{=} -\alpha N_t \Delta t + \tilde{\sigma}\sqrt{N_t}\Delta W_t. \tag{9}$$

Formally, this gives rise to the following SDE

$$dN_t = -\alpha N_t dt + \tilde{\sigma}\sqrt{N_t} dW_t. \tag{10}$$

In developing (10), the external neutron source has not been accounted for. This is addressed next. From a physical point of view it is a shot noise, emitting each time a single neutron with fixed intensity which we denote by S. Over a time interval of length $\Delta t \ll 1$, the probability of emission is given by $S\Delta t$, independently over non-intersecting intervals. This structure is known to lead to a Poisson process with rate S. Again, with ΔS_t denoting the increase in the number of emissions over the time interval $[t, t + \Delta t]$, an argument that uses the CLT as above gives rise to the approximation

$$\Delta S_t \stackrel{d}{=} S\Delta t + \sqrt{S}\Delta W_t. \tag{11}$$

Hence, modifying (9) to account for the source gives

$$\Delta N_t \stackrel{d}{=} -\alpha N_t \Delta t + S\Delta t + \sqrt{\tilde{\sigma}^2 N_t + S}\Delta W_t. \tag{12}$$

Taking a formal limit gives the SDE

$$dN_t = -\alpha N_t dt + S dt + \sqrt{\tilde{\sigma}^2 N_t + S} dW_t.$$

A further simplifying step amounts to approximating the factor N_t under the root by its steady state mean field value, S/α. From a physical point of view, this can be justified by the assumptions that the statistical fluctuations are sufficiently smaller than N_t. Clearly, this approximation is only possible if there are no rapid power shifts in the core.

Denote

$$\hat{\sigma} = \sqrt{\tilde{\sigma}^2 \frac{S}{\alpha} + S} = \sqrt{\frac{S}{\alpha}\lambda_f \overline{\nu(\nu - 1)} + 2S}, \tag{13}$$

where the last equality follows from (7). Then we have

$$dN_t = -\alpha N_t dt + S dt + \hat{\sigma} dW_t. \tag{14}$$

We will refer to Eq. (14) as the *zero power stochastic PKE*, or the SPKE, for short.

From a mathematical point of view, the SDE (14) that we have just introduced is closely related to the Fokker-Planck equation (FPE) presented in Williams (1974) in the following sense. The evolution of the probability density function of the stochastic process N_t of (14) is given by the solution to the FPE. The modeling approach on the other hand is very different. The SDE describes the neutron population dynamics directly, whereas the FPE describes its probability density function.

Remark 3.1. Notice that the coefficient α is consistent with the coefficient $\lambda - \bar{\nu}\lambda_f$ from the mean field Eq. (5), as follows from the identity (7). Thus it is natural to view (14) as a version of (5) with an additional stochastic term that models the noise.

The derivation of the SPKE was based on the CLT approximation as well as treating the mean field counterpart as a steady state. The latter assumption is relevant also for the initial condition. That is,

for the system to be at or near equilibrium, one must assume that N_0 is close to S/α, the asymptotic mean field value. Our precise assumption on the initial condition appears below in Remark 3.2.

We shall need the following standard facts about solutions to SDE of the form

$$X_t = -AX_t + Sdt + \sigma dW_t$$

where $A > 0$ and $\sigma > 0$ are constants, such as (14). If the initial condition is normal and independent of the driving BM W, then the process X is a Gaussian process, known as an Ornstein-Uhlenbeck process (Øksendal, 2003). If, specifically, the initial condition X_0 is distributed as $\mathcal{N}(S/A, \sigma^2/(2A))$ then the process is *stationary*, and one has

$$E(X_t) = \frac{S}{A}, \qquad E(X(t)^2) = \frac{\sigma^2}{2A} + \frac{S^2}{A^2}, \qquad \mathrm{Var}(X_t) = \frac{\sigma^2}{2A}. \tag{15}$$

Moreover, its *autocorrelation function*, defined by $\phi_X(\tau) = E(X_t X_{t+\tau})$, is (independent of t and) given by

$$\phi_X(\tau) = \frac{\sigma^2}{2A} e^{-A\tau}. \tag{16}$$

Remark 3.2. We shall assume that the initial condition N_0 is distributed as $\mathcal{N}(S/\alpha, \hat{\sigma}^2/(2\alpha))$, independent of the driving BM. By the preceding paragraph, this process is then a stationary Gaussian process, and formulas (15) and (16) are in force.

3.2. Coupled SDE for a model with destructive detection

Whereas the neutron population size is perhaps the most basic quantity in a nuclear system, a practical viewpoint must take into account that it cannot be measured noiselessly. There are several reasons for this. First, measuring the population size is always performed over a time window of positive duration rather than at a single epoch. Second, the detection precess itself amounts to a random sampling procedure, resulting with an inevitable sampling noise. Third, in contrast to the model assumption, the neutron population has a spatial distribution and the detection system has detectors only in designated positions, thus only a small fraction of the neutrons will be detected, and the actual fraction might be unknown. Finally, the detection process is also a destructive one, as detections are achieved by absorption of neutrons in the detector, and as a result, detection directly influences the overall population.

To model the detection count we consider it as one more reaction type. From the population dynamics point of view, the detection is nothing more than absorption. Therefore, as far as the population size is concerned, the analysis from Section 3.1 covers the case of destructive detection once one modifies the absorption intensity parameter λ_a to account for the absorption associated with detection. In what follows, we aim at deriving the evolution of the population size and detection count jointly.

Let us then denote by λ_d the probability per time unit of a neutron to create a detection reading, and by λ_ℓ the probability per time unit for an absorption of a neutron not associated with detection. Then $\lambda_a = \lambda_\ell + \lambda_d$, and $\lambda = \lambda_a + \lambda_f = \lambda_\ell + \lambda_d + \lambda_f$ gives the probability per time unit for any reaction. Thus the detection efficiency is given by λ_d/λ.

Let $\{D_t\}$ be the detection counting process, where, for each t, D_t equals the number of detections in the time interval $[0, t]$. Fix an interval $[t, t+\Delta t]$ and denote by $\Delta D_t = D_{t+\Delta t} - D_t$ the number of detections over the interval. If Δt is small then, conditionally on $N_t, \Delta D_t$ is approximately given by a binomial with parameters $(N_t, \lambda_d \Delta t)$. A CLT approximation thus gives

$$\Delta D_t - N_t \lambda_d \Delta t \overset{d}{=} \mathcal{N}(0, N_t \lambda_d \Delta t). \tag{17}$$

Let $\{W_t\}$ and $\{\widetilde{W}_t\}$ be two independent standard BM. Then we can couple the increment ΔD_t with the neutron population N_t to obtain

$$\Delta N_t \overset{d}{=} -\alpha_1 N_t \Delta t + S\Delta t + \sigma_1 \Delta W_t - \Delta D_t \tag{18}$$

$$\Delta D_t \overset{d}{=} \lambda_d N_t \Delta t + \sigma_2 \Delta \widetilde{W}_t, \tag{19}$$

where (18) is obtained from (12) by correcting for the loss due to detection, and (19) expresses (17) under a steady state assumption. Here,

$$\alpha_1 = \lambda_f + \lambda_\ell - \overline{\nu}\lambda_f,$$

where we did not include λ_d in the coefficient of $N_t \Delta t$, since the losses due to detection are accounted for in the term $-\Delta D_t$,

$$\sigma_1^2 = \frac{S}{\alpha}(\lambda_f + \lambda_\ell + \lambda_f(\overline{\nu^2} - 2\overline{\nu})) + S$$

is the variance of the contribution of all the reactions but the detections, and

$$\sigma_2^2 = \frac{S}{\alpha}\lambda_d$$

is the variance associated with detection. The latter two parameters were computed as follows. The first part of Eq. (13) gives $\hat{\sigma}$ in terms of $\tilde{\sigma}$. We used the same relation, but replaced $\tilde{\sigma}$ of (7) by taking $\lambda_f + \lambda_\ell$ in place of λ, for the same reason specified above. Note however that in the expression S/α for the mean field steady state, we kept α rather than using α_1, as the steady state value is clearly affected by the detection losses. Similarly, σ_2 was calculated based on (17), where N_t was replaced by its mean field steady state value S/α.

Once again we take formal limits. We obtain from (18) and (19) the coupled system of SDE

$$\begin{cases} dN_t = -\alpha_1 N_t dt + \sigma_1 dW_t - dD_t + Sdt, \\ dD_t = \lambda_d N_t dt + \sigma_2 d\widetilde{W}_t. \end{cases} \tag{20}$$

Remark 3.3. We can obtain an autonomous SDE for N_t from the coupled pair (20) by substituting dD_t from the second equation into the first. The equation obtained this way is

$$dN_t = -(\alpha_1 + \lambda_d)dt + Sdt + \sigma_1 dW_t - \sigma_2 d\widetilde{W}_t. \tag{21}$$

Notice that $\alpha_1 + \lambda_d = \alpha$, while

$$\begin{aligned} \sigma_1^2 + \sigma_2^2 &= \frac{S}{\alpha}(\lambda_f + \lambda_\ell + \lambda_f(\overline{\nu^2} - 2\overline{\nu})) + S + \frac{S}{\alpha}\lambda_d \\ &= \frac{S}{\alpha}(\lambda + \lambda_f(\overline{\nu^2} - 2\overline{\nu})) + S \\ &= \hat{\sigma}^2, \end{aligned} \tag{22}$$

precisely matching the coefficients of (14). We thus conclude that Eqs. (14) and (20) describe the same dynamics as far as N_t is concerned, as we have earlier anticipated.

3.3. Recovering the Feynman-Y formula

The analysis presented in this subsection is based solely on the set of SDE (20) derived in Section 3.2. Our goal is to validate (20) by using it in a computation that recovers one of the most basic formulas in reactor noise, namely the *Feynman variance to mean ratio*. Often referred to as the *Feynman-Y* function, this formula describes the ratio between the variance and the mean of the number of

counts in an interval of duration T, as a function of T (Feynman, 1945).

Evaluating the mean number of counts is done by integrating (20) in the interval $[0,t]$ and taking the expectation. Using (15), this gives

$$E(D_t) = \lambda_d E\int_0^t N_s ds = \lambda_d St/\alpha = p_d St/(1-k). \tag{23}$$

As expected, the result for the mean coincides with value found in the literature (Pazsit and Pal, 2008).

Computing the variance is a bit more complicated, and requires several steps. First, implementing Ito's formula (Øksendal, 2003), we have

$$D_t^2 = D_0 + 2\int_0^t D_s dD_s + \int_0^t (dD_s)^2, \tag{24}$$

$$D_t N_t = D_0 N_0 + \int_0^t D_s dN_s + \int_0^t N_s dD_s + \int_0^t dN_s dD_s. \tag{25}$$

Then through (20), we have

$$\begin{aligned} D_t dD_t &= D_t(\lambda_d N_t dt + \sigma_2 d\tilde{W}_t)\\ N_t dD_t &= N_t(\lambda_d N_t dt + \sigma d\tilde{W}_t)\\ D_t dN_t &= D_t(-\alpha_1 N_t dt + \sigma_1 dW_t - dD_t + Sdt)\\ (dD_t)^2 &= (\lambda_d N_t dt + \sigma_2 d\tilde{W}_t)^2 = \sigma_2^2 dt\\ dN_t dD_t &= -(dD_t)^2 = -\sigma_2^2 dt. \end{aligned}$$

The final two identities above use the so-called multiplication table of the Ito calculus, according to which $(dt)^2 = 0, dtdW_t = 0$, $dtd\tilde{W}_t = 0, dW_t d\tilde{W}_t = 0$, while $(dW_t)^2 = (d\tilde{W}_t)^2 = dt$ (see Theorems 4.1.2 and 4.2.1, especially Eq. (4.1.8), in the book Øksendal, 2003). Substituting in Eqs. (24) and (25), and using $D_0 = 0$, yields

$$E\left(D_t^2\right) = 2\lambda_d \int_0^t E(D_s N_s) ds + \sigma_2^2 t \tag{26}$$

$$E(D_t N_t) = -\alpha\int_0^t E(D_s N_s) ds + S\int_0^t E D_s ds + \lambda_d \int_0^t E N_s^2 ds - \sigma_2^2 t, \tag{27}$$

where on the second line we used the identity $\alpha_1 + \lambda_d = \alpha$. Moreover, recalling that we assume stationarity of the process N, we have from (15),

$$E\left(N_t^2\right) = \frac{\hat{\sigma}^2}{2\alpha} + \left(\frac{S}{\alpha}\right)^2 = \frac{\frac{S}{\alpha}\lambda_f \overline{\nu(\nu-1)} + 2S}{2\alpha} + \left(\frac{S}{\alpha}\right)^2.$$

Denoting $f(t) = E(D_t N_t)$, using the above identities, Eq. (27) reduces to

$$\frac{df}{dt} = -\alpha f + \lambda_d \frac{S^2}{\alpha} t + \lambda_d \left(\frac{S}{\alpha}\right)^2 + \frac{\lambda_d}{2\alpha^2} S\lambda_f \overline{\nu(\nu-1)}.$$

This equation admits an explicit solution, given by

$$f(t) = \frac{\lambda_d S}{2\alpha^3}\left(\lambda_f \overline{\nu(\nu-1)}(1-e^{-\alpha t}) + 2\alpha St\right).$$

Substituting in (26) and performing the integration, we obtain

$$E\left(D_t^2\right) = \frac{\overline{\nu(\nu-1)}\lambda_f \lambda_d^2 St}{\alpha^3} G(t) + \left(\frac{\lambda_d St}{\alpha}\right)^2 + \frac{\lambda_d St}{\alpha}, \tag{28}$$

where we denote

$$G(t) = 1 - \frac{1-e^{-\alpha t}}{\alpha t}.$$

Combining identities (23) and (28) gives

$$\begin{aligned} \mathrm{Var}(D_t) - E(D_t) &= E(D_t^2) - [E(D_t)]^2 - E(D_t)\\ &= \frac{\overline{\nu(\nu-1)}\lambda_f \lambda_d^2 St}{\alpha^3} G(t). \end{aligned} \tag{29}$$

Recalling the notation $p_d = \lambda_d/\lambda$ for the detection efficiency, $p_f = \lambda_f/\lambda$ for the fission probability as well as the identity $\alpha = \lambda(k-1)$, we may write the above relation as

$$\mathrm{Var}(D_t) - E(D_t) = \frac{St\overline{\nu(\nu-1)}p_f p_d^2}{(1-k)^3} G(t),$$

or

$$\frac{\mathrm{Var}(D_t)}{E(D_t)} - 1 = \frac{\overline{\nu(\nu-1)}p_f p_d}{(1-k)^2}\left(1 - \frac{1-e^{-\alpha t}}{\alpha t}\right). \tag{30}$$

Eq. (29) coincides with the well known Feynman-Y formula (see, for instance, Pazsit and Pal, 2008). While the derivation of this formula is usually performed using the exact model (typically by the probability generating function method, as in Dubi and Kolin (2016)), the one just presented is based on our diffusion model, obtained from CLT approximation in steady state. We regard the fact that the Feynman-Y formula is recoverable in its precise form from our model as a strong substantiation of the latter, since this formula has indeed been validated numerous times.

4. Sampling the Feynman-Y function using the mean absolute deviation

4.1. A proposed estimator

In this section we propose an alternative to the traditional way of sampling the Feynman-Y function. While the variance is usually estimated by means of evaluating the *mean squares* (MS) of the sampled detection counts, we propose to use the *mean absolute deviation* (MAD) instead. Here, for a collection of samples $\{X_n\}_{n\leqslant N}$, the estimators alluded to above are defined as

$$M_N = \frac{1}{N}\sum_{n=1}^N X_n,\ MS_N = \frac{1}{N-1}\sum_{n=1}^N (X_n - M_N)^2,\ MAD_N = \frac{1}{N}\sum_{n=1}^N |X_n - M_N|. \tag{31}$$

Clearly, M_N and MS_N are estimators for the expectation and variance, respectively. Recall that the *first absolute moment* of a RV X is defined as

$$\mu_X^{\mathrm{abs}} = E(|X - \mu_X|). \tag{32}$$

Then MAD_N is seen to be a natural estimator for μ_X^{abs}.

At the heart of the scheme to be proposed lies the diffusion approximation developed in the previous section, according to which the detection count is approximately a Gaussian process. As a result, any detection count over a finite time window is also approximated as a Gaussian RV. For a Gaussian RV, the first absolute moment and the variance are related via the formula

$$Var(X) = \frac{\pi}{2}[\mu_X^{\mathrm{abs}}]^2. \tag{33}$$

The method we propose is to estimate the variance in the Feynman-Y formula by

$$Var(X) \approx \frac{\pi}{2} MAD_N^2, \tag{34}$$

in place of the traditional estimation scheme

$$Var(X) \approx MS_N.$$

The statistics literature on the use of the MAD and MS as measures of dispersion goes many years back. There are known advantages and disadvantages to using each of them (as well as other estimators). Specifically, an advantage of MAD over MS is in the context of robust statistics (Huber, 2011), where it is known to be (in a way that can be made mathematically precise) more

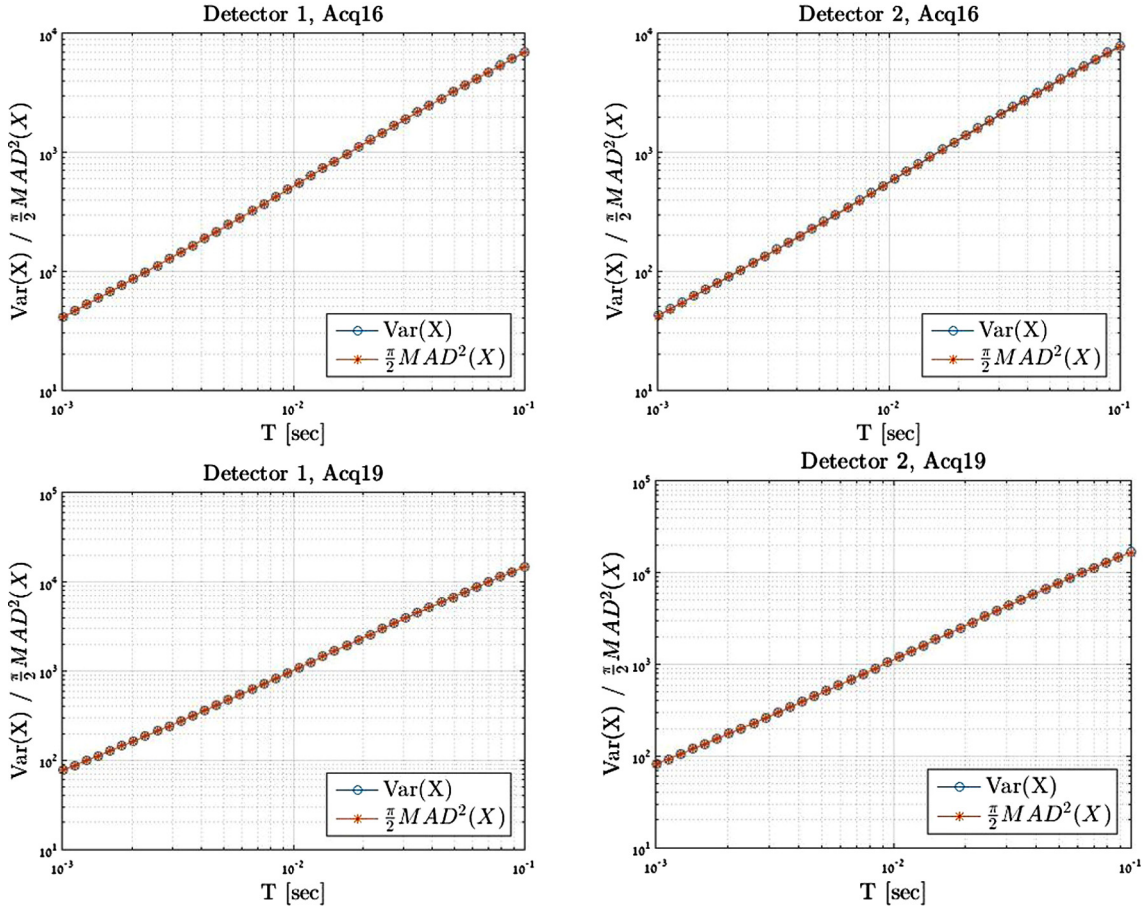

Fig. 1. Measured values for the variance and the MAD in all four signals.

robust than other estimators to perturbations in the underlying distribution, and more resistant to outliers. Our motivation to using this estimator stems from these properties.

For a general distribution, the first absolute moment is often harder to compute explicitly than the standard deviation. This is particularly true in the context of the present study: while the variance (or any other moment, for that purpose) can be expressed in terms of the probability generating function (Pazsit and Pal, 2008; Dubi and Kolin, 2016), the first absolute moment cannot. However, for the model introduced in the previous section, the Gaussianity of the distribution settles this computational aspect, for the reasons explained above. Our proposed approach thus makes full use of the Gaussian approximation that the model provides. In particular, the stochastic model (20) allows us to use the MAD in order to estimate the variance of the detection count distribution, by appealing to (34). This prediction is tested experimentally and further discussed in the next subsection.

4.2. Experimental validation for using the MAD in the neutron count distribution

In the present subsection we offer an experimental validation of the relationship in Eq. (34). The primal objective of the comparison is to further substantiate the SDE introduced in (20), which predicted the equality. In addition, we discuss some possible applications of the equality.

Validation was carried out on a set of 4 detection signals obtained during Sep. 2014, on the MINERVE reactor (Gilad et al., 2014, 2017). Measurements were taken at two (sub critical) reactivity levels, each measured in two distinct detectors (resulting with 4 detection signals). The reported reactivity of the first configuration (Acq16) was -270 pcm and -120 pcm for the second (Acq19). Both reactivity values were obtained by a rod drop experiment. Each measurement lasted approximately 1.5 h.

The MS and the MAD were both measured for a range of detection gates T ranging between $T = 10^{-3}$ [sec] and $T = 10^{-1}$ [sec], which is a typical range for the prompt neutrons when sampling the Feynman-Y variance to mean ratio. The sampling was done in consecutive windows: the detection signal was divided into N consecutive time gates of duration T, the number of detections in the nth gate ($1 \leqslant n \leqslant N$) is denoted by X_n, and the mean, the MS and MAD of the detection count distribution were computed according to (31) above. The sampled values are shown in Fig. 1.

The measured discrepancy between $MS(X)$ and $\frac{\pi}{2}MAD^2(X)$ are listed in Table (1) below.

Table 1
Discrepancy between the sampled values of $MS(X)$ and $(\pi/2)MAD^2(X)$.

Measurement	Acq 16 detector 1	Acq 16 detector 2	Acq 19 detector 1	Acq 19 detector 2
Average Difference (%)	0.09	1.06	0.07	0.07
Maximal Difference (%)	0.32	1.8	0.23	0.023

As expected, there is a very high correspondence between the sampled values of $MS(X)$ and $\frac{\pi}{2}MAD^2(X)$, forming a strong validation of equality (34) and, consequently, of the diffusion approximation (20).

The experimental results are not only a validation of a basic property of the count distribution, namely its approximate Gaussian distribution, that to the best of our knowledge has not been noticed, but might also have an impact on pile noise measurement using the Feynman-α method. As already mentioned, the approximation (34) offers an alternative way to sample the Feynman-Y function. When estimating the statistical uncertainty of the sampled value of the variance, the dominant term is proportional to the fourth moment of the detection count distribution. The variance of the $MAD(X)$, on the other hand, is obviously equal to the variance of X. Thus, the dominant term is proportional only to the second moment of the detection count distribution. Therefore, it is reasonable to ask whether sampling the Feynman-Y function using $\frac{\pi}{2}MAD^2(X)$ rather than $Var(X)$ might reduce the statistical uncertainty. We intend to address this question in future work.

5. Concluding remarks

The motivation to this work stems from the fact that the neutron population size in the core cannot be measured directly, and thus any practical approach to determining it must eventually take into account the randomness associated with detection. We thus regard modeling of the detection count distribution to be of utmost importance toward implementation of the SDE formalism in nuclear engineering and reactor core analysis. The contribution of this work is to develop the SDE approach so as to incorporate detection into the model equations. To this end, a pair of SDE is derived, coupling the detection and the neutron population. As in the basic model for neutron population, the detection process is modeled via the mean field detection rate and a noise term consisting of Brownian motion.

The model was validated in two ways. First, we analyzed the detection count distribution based on it, showing that it gives rise to the same first two moments as a calculation that uses the full stochastic transport equation. In particular, the FCLT approximation precisely preserves the first two moments of the detection count distribution. Second, we have shown experimentally that a correspondence between the MS and the MAD estimators predicted by the Gaussian property of the model, holds with high accuracy. This observation suggests a new sampling method for the Feynman-Y curve, which might be more robust, and in particular less affected by detection losses.

The authors believe that the SDE approach and its present development are promising in analyzing the neutron count distribution in far more complex settings. These include settings that account for the delayed neutrons, thermal-hydraulic (or other non-linear) feedback, as well as the effect of dead-time in the detection system. These aspects will be treated in future work.

Acknowledgment

This research was supported in part by the Israeli Science Foundation (grant 1184/16).

References

Dawson, D.A., Li, Z., 2012. Stochastic equations, flows and measure-valued processes. Ann. Probab. 40.
Demeshko, M., Washio, T., Kawahar, Y., Pepyolyshev, Y., 2016. A novel continuous and structural VAR modeling approach and its application to reactor noise analysis. ACM Transactions on Intelligent Systems and Technology (TIST)—Special Issue on Causal Discovery and Inference Archive, 7 (2), January 2016.
Diniz, R., dos Sontas, A., 2002. A noise analysis approach for measuring the decay constants and relative abundance of delayed neutrons in zero power reactors. J. Nucl. Sci. Technol. (Supplement2).
Diniz, R., dos Sontas, A., 2006. Experimental determination of the decay constants and abundances of delayed neutrons by means of reactor noise analysis. Nucl. Sci. Eng. 152, 125–142.
Dubi, C., Kolin, A., 2016. Analytic derivation of the statistical error in Feynman-α method. Ann. Nucl. Energy 88.
Ensslin, N., Harker, W.C., Krick, M.S., Langner, D.G., Pickrell, M.M., Stewart, J.E., 1998. Application Guide to Neutron Multiplicity Counting. Los Alamos National Laboratory LA–13422–M.
Ethier, S.N., Kurtz, T.G., 1986. Markov Processes, Characterization and Convergence. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York. 0-471-08186-8. x +534 pp.
Feynman, R.P., 1945. The statistics of neutron chain. Technical Report Los Alamos Report, LA571.
Gard, T.C., 1988. Introduction to stochastic differential equations. Monographs and Textbooks in Pure and Applied Mathematics, volume 114. Marcel Dekker Inc., New York.
Gilad, E., Rivin, O., Yaar, I., Ettedgui, H., Geslot, B., Pepino, A., Di Salvo, J., Gruel, A., Blaise, P., 2014. Estimation of the delayed neutron fraction β_{eff} of the MAESTRO Core in MINERVE zero power reactor. Physor International Conference, Sep. 28th–Oct. 3rd. 2014.
Gilad, E., Geslot, B., Blaise, P., Dubi, C., 2017. Sensitivity of power spectral density techniques to numerical parameters in analyzing neutron noise experiments. Progress in Nuclear Energy. In press.
Ha, P., Kim, J., 2010. A stochastic approach to monoenergetic space-time nuclear reactor kinetics. J. Nucl. Sci. Technol. 47 (8).
Ha, P., Kim, J., 2011. Numerical investigating of dynamic fluctuations following nuclear reactor transients using a simplified stochastic model. J. Nucl. Sci. Technol/ 48 (2).
Allen, E., 2013. A stochastic analysis of power doubling time for a subcritica system. Stochastic Anal. App. 31.
da Silva, M., 2016. A nonstiff solution for the stochastic neutron point kinetics equations. Ann. Nucl. Energy 97, 47–52.
Hayes, J., Allen, E., 2005. Stochastic point-kinetics equations in nuclear reactor dynamics. Ann. Nucl. Energy 32.
Huber, P.J., 2011. Robust Statistics. Springer, Berlin, Heidelberg.
Karatzas, I., Shreve, S.E., 1991. Brownian Motion and Stochastic Calculus. Springer-Verlag, New York. 0-387-97655-8. xxiv+470 pp, second edition edition.
Karlin, S., Taylor, H.M., 1981. A Second Course in Stochastic Processes. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London.
Kawazu, K., Watanabe, S., 1971. Branching processes with immigration and related limit theorems. Teor. Verojatnost. i Primenen. 16.
Malinovitch, T., Dubi, C., 2015. A multi-region multi-energy formalism for the Feynman-α formula. Ann. Nucl. Energy 76.
Øksendal, B., 2003. Stochastic Differential Equations. Springer-Verlag, Berlin.
Ott, K., Neuhold, R., 1985. Introductory Nuclear Reactor Dynamics. American Nuclear Society.
Pakes, A.G., 1971. Branching processes with immigration. J. Appl. Probab. 8.
Pazsit, I., Pal, L., 2008. Neutron Fluctuations: A Treatise on the Physics of Branching Process. Elsevier Print.
Uhrig, R., 1970. Random Noise Techniques in Nuclear Reactor Systems. RE Ronald Press.
Williams, M.M.R., 1974. Random Processes in Nuclear Reactors. Pargamon Press.