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RISK-SENSITIVE CONTROL FOR THE PARALLEL SERVER MODEL*

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Abstract. A Markovian queueing model is considered in which servers of various types work in parallel to process jobs from a number of classes at rates μ_{ij} that depend on the class, i, and the type, j. The problem of dynamic resource allocation so as to minimize a risk-sensitive criterion is studied in a law-of-large-numbers scaling. Letting $X_i(t)$ denote the number of class-i jobs in the system at time t, the cost is given by $E\exp\{n[\int_0^T h(\bar{X}(t))dt + g(\bar{X}(T))]\}$, where $T > 0$, h and g are given functions satisfying regularity and growth conditions, and $\bar{X} = \bar{X}^n = n^{-1}X(n\cdot)$. It is well known in an analogous context of controlled diffusion, and has been shown for some classes of stochastic networks, that the limit behavior, as $n \to \infty$, is governed by a differential game (DG) in which the state dynamics is given by a fluid equation for the formal limit φ of $\bar{X}$, while the cost consists of $\int_0^T h(\varphi(t))dt + g(\varphi(T))$ and an additional term that originates from the underlying large-deviation rate function. We prove that a DG of this type indeed governs the asymptotic behavior, that the game has value, and that the value can be characterized by the corresponding Hamilton–Jacobi–Isaacs equation. The framework allows for both fixed and a growing number of servers $N \to \infty$, provided $N = o(n)$.

Key words. parallel server model, risk-sensitive control, large deviations, differential games, Hamilton–Jacobi–Isaacs equation, many-server queue

AMS subject classifications. 60F10, 60K25, 49N70, 93E20

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1. Introduction.

In the parallel server model (PSM), servers of various types process jobs from a number of classes, where each job requires service exactly once. Each class can be served at least by one of the types of servers, but not necessarily by all. A natural problem is to find a dynamic resource allocation policy to minimize a cost of interest. The model has been studied extensively in recent years due to its relevance in telephone call centers and in computer data systems. A sample of references treating this problem in fluid and diffusion regimes and via dynamic programming techniques is [2], [6], [8], [9], [10], [11], [12], [13], [19], [20], [21], and [26] (see [1] for a more comprehensive list).

The operation of queueing systems so as to avoid large exceedances of queue length and waiting time, such as for buffer overflow considerations or quality of service assurance, is of prime importance in practice. A natural way to address these considerations is to associate with the model a risk-sensitive cost criterion, that heavily penalizes such large exceedances. It is well known for controlled diffusion models [15], [18] and has been shown for classes of stochastic networks [3], [4], [5], [14] that considering a law-of-large-numbers scaling with this type of criterion brings into play large deviations phenomena, due to the fact that the most significant contribution to the cost originates from atypically large perturbations of the underlying state process. The

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dynamic control problem asymptotics, which is difficult to obtain directly, can then be analyzed by a differential game (DG) associated with a perturbed fluid model. As a result, the asymptotic regime is different from the fluid or diffusion regimes. While asymptotically optimal control for stochastic networks has been extensively studied for queueing network models under fluid and diffusion regimes, analogous problems in the large deviation regime have been addressed for a very limited number of models, and to the best of our knowledge include only the papers [4] and [25] (see below). The goal of this paper is to enrich the variety of models addressed by this approach, by considering the PSM with a risk-sensitive criterion. Our results show that a DG of the type alluded to above indeed governs the asymptotic behavior, that the game has value, and that the value can be characterized by the corresponding dynamic programming equation of Hamilton–Jacobi–Isaacs (HJI) type.

The model is treated in a Markovian setting, assuming that jobs of class i arrive with a certain rate λ_i, and that the total processing capacity for class-i jobs by type-j servers is given by μ_{ij}. Denoting by $X_i(t)$ the number of class-i jobs in the system at time t, the cost is given by

$$E \exp\left\{ n \left[\int_0^T h(\bar{X}(t))dt + g(\bar{X}(T)) \right] \right\}, \tag{1.1}$$

where $T > 0$ is fixed, h and g are given functions, and $\bar{X} = \bar{X}^n = n^{-1}X(n\cdot)$. Considering such a cost for large values of n puts emphasis on large values of $\int_0^T h(\bar{X}(t))dt + g(\bar{X}(T))$. Qualitatively it is clear that the cost specified above is closely related to the risk-sensitive cost for excessive waiting time. In a model with customer abandonment from the queue, it is related to the cost that accounts for a large abandonment count. While this provides further motivation to study the problem, in this paper we do not make precise statements regarding these alternative measures of performance.

A related work is [25], analyzing a large class of non-Markovian single-server models with multiple input flows in a large deviation regime. A specific policy called *largest weighted delay first* scheduling is analyzed and proved to be asymptotically optimal for the problem of minimizing the decay rate of excessive wait probabilities.

A paper closer to the present work is [4], that studies a class of controlled stochastic networks of reentrant line structure, under a risk-sensitive cost associated with escape time (such as the time until the buffer limit is reached), establishing relations to the corresponding DG and HJI equation. A sequel [5] establishes explicit solutions in the case of a network of queues in tandem. While the network model is different, our approach is closely related to and builds on the framework of [4]. Besides the difference in the model, there are however several important aspects in which the treatments differ at the technical level. First, the fixed time horizon form of (1.1) is different from one based on exit time. Second, the unboundedness of g makes the treatment of both the DG and the HJI equation more subtle. Third, the assumptions made in [4] prohibit routing control, where a job could be handled by more than one server. In particular, a feature that makes the boundary analysis convenient in [4] is the spatial homogeneity of the controlled generator in the "interior" of the domain (i.e., when $X_i > 0$ for all i), with boundary corrections given via a fixed, continuous Skorokhod map. This is not valid for the PSM. Indeed, already in the case of a single class with two servers, there is a difference between the set of possible jump intensities when there are two or more jobs in the system (the jump rate from x to $x - 1$ could be as large as $\mu_1 + \mu_2$) and when there is only one (the jump rate to zero is at most $\mu_1 \vee \mu_2$). Although the Skorokhod map plays an important role in the present treatment, its use is less straightforward.

In a work in progress [7] we continue this line of research by considering examples where one can explicitly solve the game and show that a priority policy that mimics the DG solution is asymptotically optimal.

The organization of this paper is as follows. Section 2 introduces the model and states the first set of main results, that characterize the asymptotics in terms of the DG and HJI equation. Section 3 introduces tools required to prove these results. Section 4 addresses the stochastic control problem–DG relation, while section 5 proves the DG-PDE relation.

2. Model and main results. The model is parameterized by $n \in \mathbb{N}$. It consists of I customer classes and J service stations, where station $j \in \mathcal{J} := \{1, 2, \dots, J\}$ contains $N_j(n) \geq 1$ identical servers. While I and J are fixed, $N_j = N_j(n)$ may vary with n. Denoting $N(n) = \sum_j N_j(n)$, it is assumed that

$$\lim_{n\to\infty} \frac{N(n)}{n} = 0. \tag{2.1}$$

Note that having $N_j(n) = N_j$ fixed (independent of n) is a legitimate special case.

Arrivals into the system occur according to independent Poisson processes, denoted by E_i, $i \in \mathcal{I} := \{1, 2, \dots, I\}$, with respective parameters $\lambda_i(n)$, where

$$\lambda_i(n) = \lambda_i n, \qquad i \in \mathcal{I}, \tag{2.2}$$

and $\lambda_i > 0$ are fixed. The servers are exponential, where a class-i customer can be served at rate $\mu_{ij}(n) \geq 0$ by a server from station j. Having $\mu_{ij}(n) = 0$ is possible, and means that a server from station j is unable to serve a class-i customer. It is assumed that the total processing capacity of class-i customers by station j, namely, $N_j(n)\mu_{ij}(n)$ satisfies

$$N_j(n)\mu_{ij}(n) = \mu_{ij} n, \qquad i \in \mathcal{I},\ j \in \mathcal{J}, \tag{2.3}$$

where $\mu_{ij} \geq 0$ are fixed. Thus both the arrival rates and the per-station total service capacity scale like n.

Denote the number of class-i customers in the system at time t by $\Xi_t^{n,i}$ and write $\Xi^n = (\Xi_t^{n,i})_{i\in\mathcal{I},t\geq 0}$ for the process taking values in $\mathbb{Z}_+^I$. A normalized version is

$$X_t^n = n^{-1}\Xi_t^n, \qquad t \geq 0, \tag{2.4}$$

which is a process taking values in $G^n := n^{-1}\mathbb{Z}_+^I$. Denote $G = \mathbb{R}_+^I$ and G^o the interior of G.

Control processes will be associated with service allocation. We first describe the action space. An *allocation matrix* is any member of

$$U := \left\{ u \in \mathbb{R}_+^{I\times J} : \sum_{i\in\mathcal{I}} u_{ij} \leq 1,\ j \in \mathcal{J} \right\}. \tag{2.5}$$

If $u \in U$ and $N_j(n)u_{ij}$ is an integer for all i, j, this quantity represents the number of servers from station j allocated to serve class-i customers. For simplicity, the product $N_j(n)u_{ij}$ is not required to be integer, and thus (1) if there are multiple customer classes, a server may work on more than one job simultaneously, and (2) a server may devote less than its full capacity even if there is only one customer class. Note that in our formulation, a given processor never serves two or more customers from the same class. We have a remark about a different formulation later in Remark 4.1.

The precise formulation of control is based on the martingale approach. To describe it, introduce the *controlled generator* acting on the space of functions $G^n \to \mathbb{R}$. It is given, for each $n \in \mathbb{N}$ and $u \in U$, by

(2.6)
$$\mathcal{L}^{n,u} f(x) = \sum_{i\in\mathcal{I}} n\lambda_i \left(f\left(x + \frac{1}{n}e_i\right) - f(x)\right) + \sum_{(i,j)\in\mathcal{I}\times\mathcal{J}} n\mu_{ij}u_{ij} \left(f\left(x - \frac{1}{n}e_i\right) - f(x)\right) \mathbf{1}_{\{x-\frac{1}{n}e_i\in\mathbb{R}^I_+\}}, \qquad x \in G^n,$$

where $\{e_i\}$ denote the standard basis of $\mathbb{R}^I$. Let a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be given, supporting the processes defined below. Given n and an initial condition $(t, x) \in \mathbb{R}_+ \times G^n$, a *control system starting from* (t, x) is a triplet $S^n = (U^n, X^n, (\mathcal{F}_s)_{s\ge t})$, where U^n and X^n are processes defined on $[t, \infty)$, taking values in U and G^n, respectively, and having right continuous with finite left limits (RCLL) sample paths, $\mathcal{F}_s \subset \mathcal{F}$, $s \ge t$ forms a filtration to which these processes are adapted, and

- $\mathbb{P}(X^n_t = x) = 1$;
- one has $\mathbb{P}$-a.s.,

$$\sum_{j\in\mathcal{J}} N_j(n)U^n_{ij}(s) \le \Xi^{n,i}_s \equiv nX^{n,i}_s \qquad \text{for all } s \ge t,\ i \in \mathcal{I}; \tag{2.7}$$

- for each bounded $f : G^n \to \mathbb{R}$, the process

$$f(X^n_s) - \int_t^s \mathcal{L}^{n,U^n(r)} f(X^n_r)dr, \qquad s \ge t, \tag{2.8}$$

  is a martingale w.r.t. $(\mathcal{F}_s)_{s\ge t}$.

U^n is said to be a *control* and X^n the *associated controlled Markov process*. Given n and $(t, x) \in \mathbb{R}_+ \times G^n$, denote by $\mathcal{S}_{n,t,x}$ the corresponding class of control systems.

To present the risk-sensitive control problem, let h and g be globally Lipschitz functions from $\mathbb{R}^I_+$ to $\mathbb{R}$, monotone nondecreasing with respect to the usual partial order on $\mathbb{R}^I_+$. Further, assume that the function h is bounded. Fix $T > 0$. The cost associated with a member $S = (U^n, X^n, (\mathcal{F}^n_s))$ of $\mathcal{S}_{n,t,x}$ (where $t \in [0, T]$) is given by

$$C^n(t, x, S) = \frac{1}{n} \log \mathbb{E}[e^{n[\int_t^T h(X^n_s)ds+g(X^n_T)]}]. \tag{2.9}$$

The value function is given by

$$V^n(t, x) = \inf_{S\in\mathcal{S}_{n,t,x}} C^n(t, x, S), \qquad t \in [0, T],\ x \in G^n. \tag{2.10}$$

The first main result relates the limit of V^n, as $n \to \infty$, to a PDE of HJI type. To state it we need some notation. Set $m_0 = ((\lambda_i)_{i\in\mathcal{I}}, (\mu_{ij})_{i\in\mathcal{I},j\in\mathcal{J}})$. Then m_0 is a member of $M := \mathbb{R}^I_+ \times \mathbb{R}^{I\times J}_+$. We write generic members of M as $m = ((\bar\lambda_i)_{i\in\mathcal{I}}, (\bar\mu_{ij})_{i\in\mathcal{I},j\in\mathcal{J}})$. While λ and μ denote the actual arrival and service parameters for the system, a possibly different member m of M will be interpreted as a perturbed set of parameters. For $u \in U$ and $m \in M$, let

$$v(u,m) = \sum_i \bar{\lambda}_i e_i - \sum_{ij} u_{ij}\bar{\mu}_{ij} e_i, \tag{2.11}$$

$$\rho(u,m) = \sum_i \lambda_i \, l\Big(\frac{\bar{\lambda}_i}{\lambda_i}\Big) + \sum_{ij} u_{ij}\mu_{ij} \, l\Big(\frac{\bar{\mu}_{ij}}{\mu_{ij}}\Big), \tag{2.12}$$

where

$$l(x) = \begin{cases} x\log x - x + 1, & x \geq 0, \\ +\infty, & x < 0, \end{cases}$$

with the convention $0 \log 0 = 0$ and $l(\varepsilon/0) = \infty$ for $\varepsilon > 0$. Let

$$H(p) = \inf_{u\in U} \sup_{m\in M} [\langle p, v(u,m)\rangle - \rho(u,m)], \qquad p \in \mathbb{R}^I. \tag{2.13}$$

Let $\mathbf{I} : \partial G \to 2^{\mathcal{I}}$ be defined by

$$\mathbf{I}(x) = \{i \in \mathcal{I} : x_i = 0\}.$$

The HJI equation, considered with boundary and terminal conditions, is as follows (denoting V_t as the derivative of V w.r.t. t, and DV the gradient of V w.r.t. x):

$$\begin{cases} V_t + H(DV) + h = 0 & \text{in } [0,T) \times G^o, \\ \langle DV(t,x), e_i\rangle = 0, & x \in \partial G, i \in \mathbf{I}(x), \\ V(T,x) = g(x), & x \in G. \end{cases} \tag{2.14}$$

The precise definition of a solution to (2.14) is given in section 3.

THEOREM 2.1. *Given $t \in [0,T]$ and $G^n \ni x^n \to x \in G$,*

$$\lim_{n\to\infty} V^n(t,x^n) = V(t,x),$$

where V is the unique viscosity solution of (2.14).

While the above result characterizes the limit behavior of V^n in terms of the HJI equation, we will have an additional characterization of it as the value of a DG (Theorem 3.3).

3. Preliminaries.

We introduce the main tools on which the proof of Theorem 2.1 relies: (1) two alternative queueing models, used to bound the performance of the original model; (2) viscosity solutions of equation (2.14); (3) a differential game. At the end of this section we provide the proof of Theorem 2.1, that uses these tools, and present Theorem 3.3 regarding the relation to the DG.

3.1. Two alternative models.

The constraint (2.7) is difficult to work with directly. We introduce two models that are more convenient, not having such a constraint. They will be used to treat the original model.

Model (a). Recall the definition of the class $\mathcal{S}_{n,t,x}$. We let $\mathcal{S}^{(a)}_{n,t,x}$ be defined the same way, except that the constraint (2.7) is removed, and call the resulting model Model (a). In the original model, the total processing rate for a given class i, namely,

$\sum_j n\mu_{ij}U^n_{ij}$, can get as large as $\sum_j n\mu_{ij}$, provided $\Xi^{n,i} \ge N(n)$ (see (2.6) and (2.7)). Indeed, this is achieved by selecting $U^n_{ij} = 1$ for all j, which corresponds to a situation where class i occupies all servers in every station of the system. When $\Xi^{n,i}$ is less than $N(n)$, the maximum possible total processing rate for class i decreases in the original model, while in Model (a) it remains at the same level. A physical interpretation of Model (a) could be that multiple servers can simultaneously work on a single job, having their processing rates sum up.

As is clear from the very definition of the two models, $\mathcal{S}_{n,t,x} \subset \mathcal{S}^{(a)}_{n,t,x}$.

Model (b). $\mathcal{S}^{(b)}_{n,t,x}$ is defined by replacing (2.7) by the requirement that, for each $i \in \mathcal{I}$,

$$\Xi^{n,i} \le N(n) \quad \text{implies} \quad \sum_{j\in\mathcal{J}} U^n_{ij} = 0. \tag{3.1}$$

The physical meaning of the resulting model, called Model (b), is that one simply ceases to serve class-i customers whenever they are too few. Based on (2.5), it is clear that (3.1) implies (2.7). As a result, $\mathcal{S}^{(b)}_{n,t,x} \subset \mathcal{S}_{n,t,x}$.

The two models automatically provide bounds on the original model. That is, for $n \in \mathbb{N}$ and $t \in [0,T], x \in G^n$,

$$Q^n(t,x) := \inf_{S\in\mathcal{S}^{(a)}_{n,t,x}} C^n(t,x,S) \le V^n(t,x) \le R^n(t,x) := \inf_{S\in\mathcal{S}^{(b)}_{n,t,x}} C^n(t,x,S). \tag{3.2}$$

Models (a) and (b) are quite similar: the controlled transition rates are of the same form $\sum_j n\mu_{ij}u_{ij}$ in direction $-e_i$, for arbitrary $u \in U$, as long as the jump is permissible. Further, denote

$$\nu_n = \frac{N(n)}{n}.$$

Let $\bar\nu_n \in \mathbb{R}^I$ denote the vector $(\nu_n, \dots, \nu_n)$. Set

$$G^*_n = \{x + \bar\nu_n : x \in G^n\}, \qquad G^{\#}_n = G^n \setminus G^*_n. \tag{3.3}$$

Note that under Model (b), if X^n starts in G^*_n it will never leave this set. Indeed, if $X^n(t)$ is in G^*_n but $X^n_i(t) = \bar\nu_n$ for some i then $\Xi^{n,i}(t) = N(n)$; hence by (3.1) $U^n_{ij}(t) = 0$ for all j. By the form of the generator (2.6), all terms with $f(x - n^{-1}e_i) - f(x)$, which correspond to a downward jump, vanish. This means that where X^n is in that position it will not make a downward transition, and will remain in G^*_n. This is analogous to the fact that under Model (a) X^n never leaves G^n.

The following useful estimates on these models are proved in section 4. Throughout, $\|\cdot\|$ denotes the Euclidean norm.

LEMMA 3.1. *There exists a constant c_0 such that, for all n, $t \in [0,T]$,*

$$|Q^n(t,x) - Q^n(t,x')| \le c_0\|x - x'\|, \qquad x, x' \in G^n, \tag{3.4}$$

$$|R^n(t,x) - R^n(t,x')| \le c_0\|x - x'\|, \qquad x, x' \in G^*_n, \tag{3.5}$$

$$R^n(t,x) \le R^n(t, x + \bar\nu_n) + c_0\nu_n, \qquad x \in G^{\#}_n, \tag{3.6}$$

$$R^n(t_1,x) \le R^n(t,x) + c_0(t_1 - t), \qquad x \in G^*_n,\ t_1 \in (t,T]. \tag{3.7}$$

3.2. Viscosity solutions. Solutions to (2.14) are defined in the viscosity sense.

DEFINITION 3.2. *Let $V : [0,T] \times G \to \mathbb{R}$ be continuous in the first variable, uniformly over $[0,T] \times G$, and satisfy a global Lipschitz condition in the second, namely,*

$$\sup\{|x-y|^{-1}|V(t,x)-V(t,y)| : t \in [0,T], x \neq y \in G\} < \infty.$$

Then V is said to be a sub (super) solution of (2.14) if $V(T,\cdot) = g$, and, whenever $\theta \in C^\infty$ and $V - \theta$ has a local maximum (minimum) at $(t,x) \in [0,T) \times G$, the following hold:

$$[\theta_t(t,x) + H(D\theta(t,x)) + h(x)] \vee \max_{i \in \mathbf{I}(x)} \langle D\theta(t,x), e_i \rangle \geq 0,$$
$$\Big([\theta_t(t,x) + H(D\theta(t,x)) + h(x)] \wedge \min_{i \in \mathbf{I}(x)} \langle D\theta(t,x), e_i \rangle \leq 0\Big).$$

A function is said to be a *solution* if it is both a subsolution and a supersolution.

PROPOSITION 3.1. *Let u be a subsolution and v be a supersolution. Then $u \leq v$.*

Note that this result, proved in section 5, gives uniqueness of solutions.

3.3. A differential game. Fix $T > 0$. Given $t \in [0,T)$, denote by $\mathbb{D}([t,T];\mathbb{R}^k)$ the space of RCLL functions mapping $[t,T]$ to $\mathbb{R}^k$. The *one-dimensional Skorokhod map $\Gamma_1 = \Gamma_1^{t,T}$* from $\mathbb{D}([t,T]:\mathbb{R})$ to itself is given by

$$\Gamma_1[\psi](s) = \psi(s) - \inf_{r \in [t,s]} \psi(r) \wedge 0, \qquad s \in [t,T]. \tag{3.8}$$

Let $\Gamma = \Gamma^{t,T}$ mapping $\mathbb{D}([t,T]:\mathbb{R}^I)$ to itself be given by

$$\Gamma[\psi]_i = \Gamma_1[\psi_i] \qquad \text{for } i \leq I. \tag{3.9}$$

Γ is often called the *Skorokhod map on G with normal constraint*. It is clear from the definition that, for $\psi, \phi \in \mathbb{D}([t,T];\mathbb{R}^I)$,

$$\sup_{[t,T]} \|\Gamma[\psi] - \Gamma[\phi]\| \leq 2 \sup_{[t,T]} \|\psi - \phi\|. \tag{3.10}$$

Let

$$\bar{U} = \{u : [0,T] \to U; u \text{ is measurable}\},$$
$$\bar{M} = \{m : [0,T] \to M; m \text{ is measurable}, l \circ m \text{ is locally integrable}\}.$$

We describe a deterministic two-player zero-sum differential game where one player attempts to minimize a cost c (yet to be defined) by selecting a member of $\bar{U}$, corresponding to service allocation, and the other one chooses a member of $\bar{M}$, interpreted as perturbed arrival and service rates, to maximize c. To this end, consider the dynamics of the game,

$$\begin{cases} \psi(s) = x + \int_t^s v(u(r), m(r))dr, \quad s \in [t,T], \\ \varphi = \Gamma[\psi]. \end{cases} \tag{3.11}$$

Let the cost be defined by

$$c(t,x,u,m) = \int_t^T [h(\varphi(s)) - \rho(u(s), m(s))]ds + g(\varphi(T)), \tag{3.12}$$

where $\varphi = \varphi(\cdot; t, x, u, m)$ is given by (3.11). Note that neither the dynamics nor the cost are affected by the value of the controls u and m on $[0, t)$.

To define the game in the sense of Elliott and Kalton [16], we consider the notion of strategies. To this end, we endow $\bar{U}$ and $\bar{M}$ with the metric $d(v_1, v_2) = \int_0^T \|v_1(t) - v_2(t)\| dt$, and with the corresponding Borel σ-fields. A mapping $\alpha : \bar{M} \to \bar{U}$ is called a *strategy for the minimizing player* if it is measurable and if for every $m, \tilde{m} \in \bar{M}$ and $s \in [0, T]$,

$$m(r) = \tilde{m}(r) \qquad \text{for a.e. } r \in [0, s] \qquad \text{implies} \qquad \alpha[m](r) = \alpha[\tilde{m}](r) \qquad \text{for a.e. } r \in [0, s].$$

In a similar way a *strategy for the maximizing player* is defined by a mapping $\beta : \bar{U} \to \bar{M}$. The set of all strategies for the minimizing (respectively, maximizing) player will be denoted by A (respectively, B). The upper value for the game is defined as

$$V^+(t, x) = \sup_{\beta \in B} \inf_{u \in \bar{U}} c(t, x, u, \beta[u]),$$

and the lower value as

$$V^-(t, x) = \inf_{\alpha \in A} \sup_{m \in \bar{M}} c(t, x, \alpha[m], m).$$

The game is said to have value if the value functions V^+ and V^- coincide. The game is related to the stochastic control problem on the one hand, and to the PDE on the other hand, by the following two results.

PROPOSITION 3.2. *Fix $x \in G$ and $t \in [0, T]$. Then*

$$\limsup_{n \to \infty} R^n(t, x^n) \le V^-(t, x) \qquad \text{if} \qquad G_n^* \ni x^n \to x, \tag{3.13}$$

and

$$\liminf_{n \to \infty} Q^n(t, x^n) \ge V^+(t, x) \qquad \text{if} \qquad G^n \ni x^n \to x. \tag{3.14}$$

PROPOSITION 3.3. *Both V^+ and V^- are solutions of* (2.14).

Propositions 3.2 and 3.3 are proved in sections 4 and 5, respectively.

3.4. Proof of main results.

Proof of Theorem 2.1. First, note that Propositions 3.1 and 3.3 imply that the game has value, and that the value function $V := V^+ = V^-$ uniquely solves the PDE (2.14). Next, fix $t \in [0, T]$ and $x \in G$. To prove the theorem, it suffices to consider only sequences of the form $G_n^* \ni x^n \to x$ and $G_n^\# \ni x^n \to x$. In the former case, the combination of (3.2) and Proposition 3.2 shows

$$\lim V^n(t, x^n) = V(t, x),$$

as required. Consider now the case $G_n^\# \ni x^n \to x$. By (3.2) and (3.14), we still have a lower bound of the form $V(t, x)$. While (3.13) does not directly apply as an upper bound, its combination with (3.2) and (3.6), noting that $y_n := x^n + \bar{\nu}_n \in G_n^*$ and $\nu_n \to 0$, gives

$$\limsup V^n(t, x^n) \le \limsup R^n(t, y_n) \le V(t, x).$$

This completes the proof of the theorem. $\square$

As a consequence, we obtain an alternative characterization of the asymptotic behavior of V^n.

THEOREM 3.3. *For $G^n \ni x^n \to x \in G$ and $t \in [0,T]$, $\lim_{n\to\infty} V^n(t,x^n) = V(t,x)$, where V is the value of the DG.*

4. The stochastic control problem and the differential game. In this section we prove Lemma 3.1 and Proposition 3.2.

We begin with a result showing that the DG's value functions do not vary upon truncating the space M. For $b > 0$, denote

$$M_b = \{m = (\bar\lambda_i, \bar\mu_{ij}) \in M : \bar\lambda_i \le b, \bar\mu_{ij} \le b, i \in \mathcal{I}, j \in \mathcal{J}\},$$

and let $\bar M_b$ be defined as $\bar M$, with M replaced by M_b. Let also B_b be defined similarly to B, with $\bar M$ replaced by $\bar M_b$. Thus a strategy $\beta \in B_b$ maps $\bar U$ into $\bar M_b$. Finally, analogously to V^+ and V^-, set

$$V^{b,+}(t,x) = \sup_{\beta\in B_b} \inf_{u\in\bar U} c(t,x,u,\beta[u]),$$

$$V^{b,-}(t,x) = \inf_{\alpha\in A} \sup_{m\in\bar M_b} c(t,x,\alpha[m],m).$$

LEMMA 4.1. *For sufficiently large b*

$$V^{b,\pm}(t,x) = V^{\pm}(t,x) \qquad \text{for all } (t,x) \in [0,T]\times G. \tag{4.1}$$

Proof. Without loss of generality we set $t = 0$. We only prove $V^{b,+}(0,x) = V^+(0,x)$ (for b sufficiently large), because the proof regarding V^- is similar. Recall that B_b denotes the set of strategies of perturbed rates whose components are all bounded above by the constant b. Denote by $B_{b,\mu}, B_{b,\lambda} \subset B$ the sets of strategies of perturbed rates whose service and, respectively, arrival components are always bounded above by b. Thus $B_b = B_{b,\mu} \bigcap B_{b,\lambda}$.

Corresponding to a given $m \in \bar M$, we construct a specific truncation $\tilde m$, where only the service rates are truncated in the following way: $\tilde\mu_{ij} := \bar\mu_{ij} \wedge b$ for all i,j, whereas $\tilde\lambda_i = \bar\lambda_i$. Given $m \in \bar M$ and $u \in \bar U$, denote by φ and $\tilde\varphi$ the state dynamics for (m,u) and, respectively, $(\tilde m,u)$. Using vector relation $v(u(t),m(t)) \le v(u(t),\tilde m(t))$ for all $t\in[0,T]$, we have $\frac{d}{dt}\psi(t) \le \frac{d}{dt}\tilde\psi(t)$ (in the usual partial order on $\mathbb{R}^I$). It follows directly from the definition of Γ_1 (3.8) that if ψ_1 and $\tilde\psi_1$ are two one-dimensional trajectories that agree at 0 and satisfy $\psi_1(t)-\psi_1(s) \ge \tilde\psi_1(t)-\tilde\psi_1(s)$ for all $0\le s\le t\le T$, then $\Gamma_1[\psi_1] \ge \Gamma_1[\tilde\psi_1]$ on $[0,T]$. Using this monotonicity property, $\varphi(t) \le \tilde\varphi(t)$. Note that for b sufficiently large,

$$\rho(u(t),m(t)) \ge \rho(u(t),\tilde m(t)), \qquad t\in[0,T].$$

Given $\beta \in B$, let $\tilde\beta \in B_{b,\mu}$ denote the corresponding modification of β. Since h and g are nondecreasing functions, by the above analysis we obtain $c(0,x,u,\beta[u]) \le c(0,x,u,\tilde\beta[u])$. Thus

$$\begin{aligned}\sup_{\beta\in B_{b,\lambda}} \inf_{\bar U} c(0,x,u,\beta[u]) &\le \sup_{\beta\in B_{b,\lambda}} \inf_{\bar U} c(0,x,u,\tilde\beta[u])\\ &= \sup_{\beta\in B_{b,\lambda}\bigcap B_{b,\mu}} \inf_{\bar U} c(0,x,u,\beta[u]) = V^{b,+}(0,x).\end{aligned}$$

Hence, to show $V^+ = V^{b,+}$ it is sufficient to prove that

$$V^+(0,x) \le \sup_{\beta\in B_{b,\lambda}} \inf_{\bar U} c(0,x,u,\beta[u]). \tag{4.2}$$

Select b larger, if necessary, so that $b \ge b^* := \max_{i\le I} \lambda_i e^L > 0$, where $L = C_\Gamma(TC_h + C_g)$. This assures $\rho_i'(u,m)|_{\bar\lambda_i=b*} \ge L$ for all i, where we denote $\rho_i'(u,m) = \frac{\partial}{\partial\lambda_i}\rho(u,m) = \log(\bar\lambda_i/\lambda_i)$.

We use the same notation, $\tilde m$, to specify a different modification of m, where now only the arrival rates are truncated as in $\tilde\lambda_i := \bar\lambda_i \wedge b$. We continue to use $\tilde\psi$ and $\tilde\varphi$ for the corresponding state dynamics. Given u and m,

$$\begin{aligned}&\int_0^T h(\varphi(t)) - h(\tilde\varphi(t))dt + g(\varphi(T)) - g(\tilde\varphi(T))\\ &\quad\le C_h\int_0^T \|\varphi(t)-\tilde\varphi(t)\|dt + C_g\|\varphi(T)-\tilde\varphi(T)\|\\ &\quad\le L\sum_i\int_0^T(\bar\lambda_i(t)-\tilde\lambda_i(t))dt. \end{aligned}\tag{4.3}$$

By convexity of $m \mapsto \rho(u,m)$, we have for all t

$$\rho(u(t),m(t)) - \rho(u(t),\tilde m(t)) \ge \sum_i \rho_i'(u(t),\tilde m(t))(\bar\lambda_i(t)-\tilde\lambda_i(t)).$$

Note that $\bar\lambda_i(t) - \tilde\lambda_i(t)$ is nonnegative, and when it is positive one has $\tilde\lambda_i(t) = b \ge b^*$. Hence, due to the assigned value of b^*, the ith term in (4.4) is bounded below by $L(\bar\lambda_i(t) - \tilde\lambda_i(t))$. Integrating and using (4.3) gives $c(0,x,u,\beta[u]) \le c(0,x,u,\tilde\beta[u])$. Hence

$$\begin{aligned}V^+(0,x) &= \sup_{\beta\in B}\inf_{\bar U} c(0,x,u,\beta[u]) \le \sup_{\beta\in B}\inf_{\bar U} c(0,x,u,\tilde\beta[u])\\ &= \sup_{\beta\in B_{b,\lambda}}\inf_{\bar U} c(0,x,u,\beta[u]).\end{aligned}$$

This gives (4.2) and completes the proof. ◻

Next we prove Lemma 3.1. The proof uses a controlled generator similar to (2.6), for a process that need not be constrained to G^n but lives in $n^{-1}\mathbb{Z}^I$, namely,

$$\begin{aligned}\mathcal{L}_0^{n,u} f(x) &= \sum_{i\in\mathcal{I}} n\lambda_i\left(f\left(x+\frac{1}{n}e_i\right)-f(x)\right)\\ &\quad+\sum_{(i,j)\in\mathcal{I}\times\mathcal{J}} n\mu_{ij}u_{ij}\left(f\left(x-\frac{1}{n}e_i\right)-f(x)\right),\qquad x\in G^n,\end{aligned}\tag{4.4}$$

for $f : n^{-1}\mathbb{Z}^I \to \mathbb{R}$.

Proof of Lemma 3.1. We first prove (3.4). Fix (t,x,x'). Fix also $S = (U^n, X^n, (\mathcal{F}_s^n)) \in \mathcal{S}_{n,t,x}^{(a)}$. One can augment the probability space and construct a process Y^n, a filtration $\bar{\mathcal{F}}_s^n$ containing $\mathcal{F}_s^n$, such that $\mathbb{P}(Y_t^n = x) = 1$, $X^n = \Gamma[Y^n]$, and for each bounded $f : n^{-1}\mathbb{Z}^I \to \mathbb{R}$,

$$f(Y_s^n) - \int_t^s \mathcal{L}_0^{n,U^n(r)} f(Y_r^n)dr,\qquad s\ge t,$$

is a martingale w.r.t. $(\bar{\mathcal{F}}_s)$. Here and below, with an abuse of notation, we keep the symbol $\mathbb{P}$ for the measure on the larger probability space, $\mathbb{E}$ for the corresponding expectation, and (X^n, U^n) for the processes constructed on the new probability space (which have the same law as (X^n, U^n) under the original space). The expression in (2.8) continues to be a martingale on the larger filtration $(\bar{\mathcal{F}}_s)$. Such a construction, that uses additional exponential clocks for jumps of Y^n that occur when X^n is on the boundary, is standard (see, e.g., the proof of Lemma 6 of [4] for a related argument), and we omit the details.

Next let us construct on the new filtration a controlled process X'^n starting from x', simply by setting $X'^n = \Gamma[x' - x + Y^n]$. Then one directly verifies by the properties of Y^n and the Skorokhod map, that $S' = (X'^n, U^n, (\bar{\mathcal{F}}^n_s)) \in \mathcal{S}^{(a)}_{n,t,x'}$. Now, using (3.10), $\sup_{s\in[t,T]} \|X'^n(s) - X^n(s)\| \le 2\|x - x'\|$, and therefore by (2.9),

$$\begin{aligned}
&C^n(t,x,S) - C^n(t,x',S') \\
&\quad = \frac{1}{n}\log \mathbb{E}[e^{n[\int_t^T h(X^n_s)ds + g(X^n_T)]}] - \frac{1}{n}\log \mathbb{E}[e^{n[\int_t^T h(X'^n_s)ds + g(X'^n_T)]}] \\
&\quad \le \frac{1}{n}\log \mathbb{E}[e^{n[\int_t^T h(X'^n_s)ds + g(X'^n_T) + c_0\|x-x'\|]}] - \frac{1}{n}\log \mathbb{E}[e^{n[\int_t^T h(X'^n_s)ds + g(X'^n_T)]}] \\
&\quad \le c_0\|x - x'\|,
\end{aligned}$$

where the global Lipschitz property of h and g is used on the second line. Taking the infimum over $S \in \mathcal{S}^{(a)}_{n,t,x}$ shows $Q^n(t,x) \le Q^n(t,x') + c_0\|x - x'\|$, and the result (3.4) follows.

Toward proving (3.5), the following simple relation between Q^n and R^n will be useful. For any given pair $(\tilde{h}, \tilde{g})$, write $Q^n(t,x,\tilde{g},\tilde{h})$ for the value function Q^n of (3.2) where, in the cost function C^n (2.9), one replaces h and g by $\tilde{h}$ and $\tilde{g}$, respectively. Then

$$R^n(t, x + \bar{\nu}_n) = Q^n(t, x; g(\cdot + \bar{\nu}_n), h(\cdot + \bar{\nu}_n)), \qquad t \in [0,T],\ x \in G^n. \tag{4.5}$$

To obtain this identity, we will argue by correspondences between members of $\mathcal{S}^{(a)}_{n,t,x}$ and members of $\mathcal{S}^{(b)}_{n,t,x+\bar{\nu}_n}$. To this end, we first make the following observation. Recall that the way Model (a) is defined does not put any constraint on the process U^n (taking values in U), whereas under Model (b), (3.1) must be satisfied. Now, the form of the generator $\mathcal{L}^{n,u}$ (2.6) is such that whenever $x_i = 0$ for some i, the value of u_{ij}, $j \in \mathcal{J}$ is immaterial. Hence given any control system $(U^n, X^n, (\mathcal{F}_s)) \in \mathcal{S}^{(a)}_{n,t,x}$, we may assume without loss of generality, that, for each $i \in \mathcal{I}$ and a.e. s, $\sum_j U^n_{ij}(s) = 0$ if $X^n_i(s) = 0$. With this at hand, given a control system $S = (U^n, X^n, (\mathcal{F}_s)) \in \mathcal{S}^{(a)}_{n,t,x}$ starting from (t,x), $(U^n, X^n + \bar{\nu}_n, (\mathcal{F}_s))$ is a control system satisfying (3.1), and therefore $S \in \mathcal{S}^{(b)}_{n,t,x+\bar{\nu}_n}$ holds. As a result, using the correspondence $Z^n = X^n + \bar{\nu}_n$,

$$\begin{aligned}
Q^n(t,x;g(\cdot+\bar{\nu}_n),h(\cdot+\bar{\nu}_n)) &= \inf_{(U^n,X^n,(\mathcal{F}_s))\in\mathcal{S}^{(a)}_{n,t,x}} \frac{1}{n}\log \mathbb{E}[e^{n[\int_t^T h(X^n_s+\bar{\nu}_n)ds + g(X^n_T+\bar{\nu}_n)]}] \\
&\ge \inf_{(U^n,Z^n,(\mathcal{F}_s))\in\mathcal{S}^{(b)}_{n,t,x+\bar{\nu}_n}} \frac{1}{n}\log \mathbb{E}[e^{n[\int_t^T h(Z^n_s)ds + g(Z^n_T)]}] \\
&= R^n(t, x + \bar{\nu}_n).
\end{aligned}$$

On the other hand, given $S \in \mathcal{S}^{(b)}_{n,t,x+\bar{\nu}_n}$, by the definitions of the two models one has

$$(U^n, X^n, (\mathcal{F}_s)) \in \mathcal{S}^{(a)}_{n,t,x},$$

and the reverse inequality follows. This gives (4.5).

Equipped with the above identity, the claim (3.5), regarding R^n, follows from the estimate just obtained on Q^n.

The proof of (3.6) is similar to that of (3.4), and thus omitted.

Finally, we prove (3.7). Fix $x \in G_n^*$ and $0 \le t < t_1 \le T$. The function R^n, defined as the value of a control problem (3.2), satisfies the dynamic programming principle (see, e.g., section 8 of [27]), namely,

$$R^n(t,x) = \inf \frac{1}{n} \log \mathbb{E}[e^{n[\int_t^{t_1} h(X_s^n)ds + R^n(t_1, X_{t_1}^n)]}],$$

where, as in (3.2), the infimum is over $S \in \mathcal{S}_{n,t,x}^{(b)}$. Given $\varepsilon > 0$, let S and the corresponding controlled process X^n be such that

$$R^n(t,x) + \varepsilon \ge \frac{1}{n} \log \mathbb{E}[e^{n[\int_t^{t_1} h(X_s^n)ds + R^n(t_1, X_{t_1}^n)]}].$$

Denote by $-c_1$ a lower bound on h. Using Jensen's inequality,

$$R^n(t,x) + \varepsilon \ge -c_1(t_1 - t) + \mathbb{E}[R^n(t_1, X_{t_1}^n)].$$

Hence

$$R^n(t_1,x) - R^n(t,x) \le \varepsilon + c_1(t_1 - t) - \mathbb{E}[R^n(t_1, X_{t_1}^n) - R^n(t_1,x)].$$

The jump intensities of X^n are bounded, uniformly over all control systems S, by $c_2 n$, where c_2 does not depend on S, n, t_1, ε. Hence the number of jumps that X^n performs over $[t, t_1]$ is dominated by a Poisson random variable (r.v.) of mean $c_2 n(t_1 - t)$, whereas the size of each jump is $1/n$. Along with the estimate (3.5) (and recalling that from $x \in G_n^*$ the process can only jump to sites in G_n^*), this shows that

$$|\mathbb{E}[R^n(t_1, X_{t_1}^n) - R^n(t_1,x)]| \le c_0 \mathbb{E}[\|X_{t_1}^n - x\|] \le c_0 \frac{1}{n} c_2 n(t_1 - t) = c_0 c_2 (t_1 - t).$$

We obtain

$$R^n(t_1,x) - R^n(t,x) \le \varepsilon + (c_1 + c_0 c_2)(t_1 - t).$$

Sending $\varepsilon \to 0$, the result follows. $\square$

The value function Q^n, defined in terms of the cost C^n (2.9), clearly satisfies a certain dynamic programming equation (DPE) of Bellman type on $[0,T] \times G^n$. It will be more convenient, however, to take advantage of the fact that, owing to the logarithmic transformation appearing in (2.9), Q^n also satisfies a DPE of Isaacs type, corresponding to the value of a game. In this game, an additional player is introduced, controlling the transition rates. By considering this equation we follow the approach of [18], see also [17, Chapter VI].

To this end, we introduce the following two controlled generators. They are similar to (2.6) and (4.4), but now $m = (\bar\lambda, \bar\mu)$ is also controlled, namely,

$$\mathcal{L}^{n,u,m} f(x) = \sum_{i \in \mathcal{I}} n\bar\lambda_i \left(f\left(x + \frac{1}{n} e_i \right) - f(x) \right) + \sum_{(i,j) \in \mathcal{I} \times \mathcal{J}} n\bar\mu_{ij} u_{ij} \left(f\left(x - \frac{1}{n} e_i \right) - f(x) \right) \mathbf{1}_{\{x - \frac{1}{n} e_i \in \mathbb{R}_+^I\}}$$

for $f : G^n \to \mathbb{R}$, and

$$\mathcal{L}_0^{n,u,m} f(x) = \sum_{i\in\mathcal{I}} n\bar{\lambda}_i \left(f\left(x + \frac{1}{n} e_i\right) - f(x)\right) + \sum_{(i,j)\in\mathcal{I}\times\mathcal{J}} n\bar{\mu}_{ij} u_{ij} \left(f\left(x - \frac{1}{n} e_i\right) - f(x)\right)$$

for $f : n^{-1}\mathbb{Z}^I \to \mathbb{R}$.

LEMMA 4.2. *The function $Q^n : [0,T] \times G^n \to \mathbb{R}$ is continuously differentiable in t for every x, and satisfies the following Isaacs equation,*

$$\begin{cases} \inf_{u\in U} \sup_{m\in M} (\mathcal{L}^{n,u,m} Q^n(t,x) + \frac{d}{dt} Q^n(t,x) + h(x) - \rho(u,m)) = 0, \\ Q^n(T,x) = g(x). \end{cases} \tag{4.6}$$

Proof. We consider the following set of differential equations,

$$\inf_{u\in U} \mathcal{L}^{n,u} W^n(t,x) + \frac{d}{dt} W^n(t,x) + nh(x) W^n(t,x) = 0, \qquad W^n(T,x) = e^{ng(x)}. \tag{4.7}$$

The existence of a solution of the above equation follows from standard iteration techniques. Let W^n be a solution. Given any $\varepsilon > 0$, from (4.7) it follows, there is a $u^\varepsilon(r,x) \in U$ such that

$$\varepsilon > \mathcal{L}^{n,u^\varepsilon(r,x)} W^n(r,x) + \frac{d}{dr} W^n(r,x) + nh(x) W^n(r,x) \geq 0, \tag{4.8}$$

where $u^\varepsilon : [0,T] \times G^n \to \mathbb{R}$ can be selected as measurable. Now we consider a control $S^\varepsilon = (\bar{u}^\varepsilon, X^n, (\mathcal{F}_s^n)) \in \mathcal{S}_{n,t,x}^{(a)}$, where $\bar{u}^\varepsilon(r) = u^\varepsilon(r, X^n(r))$. Using Itô's formula we have that

$$\begin{aligned} &e^{n\int_t^s h(X^n(r))dr} W^n(s, X^n(s)) - W^n(t,x) \\ &\quad - \int_t^s \left[nh(X^n(r)) e^{n\int_t^r h(X^n(\tau))d\tau} W^n(r, X^n(r)) \right] dr \\ &\quad - \int_t^s \left[e^{n\int_t^r h(X^n(\tau))d\tau} \left(\frac{d}{dr} W^n(r, X^n(r)) \right.\right. \\ &\qquad\qquad \left.\left. + \mathcal{L}^{n,\bar{u}^\varepsilon(r)} W^n(r, X^n(r)) \right) \right] dr, \qquad s \in [t,T], \end{aligned}$$

is an $(\mathcal{F}_s^n)$ martingale. By substituting $s = T$ and taking the expectation of the above and using (2.9) and (4.8) we get

$$(T-t)\varepsilon > e^{nC^n(t,x,S^\varepsilon)} - W^n(t,x) \geq 0. \tag{4.9}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$W^n(t,x) = \inf_{S\in\mathcal{S}_{n,t,x}^{(a)}} e^{nC^n(t,x,S)} = e^{nQ^n(t,x)}. \tag{4.10}$$

Therefore, Q^n satisfies the terminal condition of (4.6) and also

$$0 = \inf_{u\in U} \mathcal{L}^{n,u} e^{nQ^n(t,x)} + \frac{d}{dt} e^{nQ^n(t,x)} + nh(x) e^{nQ^n(t,x)}.$$

For notational simplicity, write $\pi(x,v) := v\mathbf{1}_{nx+v\in\mathbb{Z}_+^I}$, $\delta_i f(t,x) := f(t,x+\frac{1}{n}e_i) - f(t,x)$, and $\tilde\delta_i f(t,x) := f(t,x+\frac{1}{n}\pi(x,-e_i)) - f(t,x)$ for $x\in G^n, v\in\mathbb{Z}^I$. Using these operators the above equation can be simplified as

$$0 = \inf_{u\in U}\Bigg(n\sum_i \lambda_i(e^{\delta_i Q^n(t,x)}-1) + n\sum_{ij} u_{ij}\mu_{ij}(e^{\tilde\delta_i Q^n(t,x)}-1) + n\frac{d}{dt}Q^n(t,x) + nh(x)\Bigg). \tag{4.11}$$

Now use the Legendre transformation of l (defined in section 2), namely, the identity

$$e^y - 1 = \sup_{x>0}(xy - l(x))$$

to obtain

$$\begin{aligned} 0 &= \inf_{u\in U}\sup_{(\bar\lambda,\bar\mu)}\Bigg(\sum_i \bar\lambda_i\delta_i Q^n(t,x) + \sum_{ij} u_{ij}\bar\mu_{ij}\tilde\delta_i Q^n(t,x) \\ &\qquad + \frac{d}{dt}Q^n(t,x) + h(x) - \rho(u,(\bar\lambda,\bar\mu))\Bigg) \\ &= \inf_{u\in U}\sup_{m\in M}\left(\mathcal{L}^{n,u,m}Q^n(t,x) + \frac{d}{dt}Q^n(t,x) + h(x) - \rho(u,m)\right). \qquad \square \end{aligned}$$

To state the following lemma, we need some additional notation. Recall the definition of Γ based on the one-dimensional Skorokhod map (3.8). We define a family of Skorokhod maps Γ^n, each mapping $\mathbb{D}([t,T]:\mathbb{R}^I)$ to itself, by

$$\Gamma^n[\psi]_i := \Gamma_1[\psi_i - \nu_n] + \nu_n, \qquad i\le I.$$

In fact, Γ^n is the Skorokhod map on $G+\bar\nu_n \equiv \{x\in\mathbb{R}^I : x_i\ge\nu_n, i\le I\}$, with normal constraint. Note that

$$\|\Gamma^n[\psi]-\Gamma[\psi]\| \le I\nu_n, \qquad \psi\in\mathbb{D}([t,T]:\mathbb{R}^I). \tag{4.12}$$

For $f : G_n^* \to \mathbb{R}$, denote

$$\begin{aligned} \tilde{\mathcal{L}}^{n,u,m}f(x) &= n\sum_i \bar\lambda_i\left(f\left(x+\frac{1}{n}e_i\right) - f(x)\right) \\ &\quad + n\sum_{ij}\bar\mu_{ij}u_{ij}\left(f\left(x-\frac{1}{n}e_i\right)-f(x)\right)\mathbf{1}_{\{x-\frac{1}{n}e_i\in G_n^*\}}, \\ &\quad x\in G_n^*,\ u\in U,\ m\in M. \end{aligned} \tag{4.13}$$

The proof of the following result is similar to that of Lemmas 7 and 8 of [4], and is thus omitted.

LEMMA 4.3. *Fix n, $t\in[0,T)$, and $b>0$.*

i. *Fix $x\in G^n$. Let a measurable function $u : [t,T]\times G^n\to U$ and a strategy $\beta\in B_b$ be given. Then there exists a filtered probability space $(\bar\Omega,\bar{\mathcal{F}},\{\bar{\mathcal{F}}_s\}_{[t,T]},\bar{\mathbb{P}}^n)$, and $(\bar{\mathcal{F}}_s)$-adapted RCLL processes $\bar X$, $\bar Y$, $\bar m$, and $\bar u$, taking values in G^n, $n^{-1}\mathbb{Z}^I$,*

M, and U, respectively, such that $\bar{\mathbb{P}}^n$-a.s., $\bar m = \beta[\bar u]$, $\bar u(s) = u(s, \bar X(s))$, $s \in [t,T]$, $\bar X = \Gamma(\bar Y)$, $\bar X(t) = \bar Y(t) = x$, and

$$f(s,\bar X(s)) - \int_t^s \left(\mathcal{L}^{n,\bar u(r),\bar m(r)} f(r,\bar X(r)) + \frac{\partial}{\partial r} f(r,\bar X(r))\right) dr \tag{4.14}$$

and

$$f(s,\bar Y(s)) - \int_t^s \left(\mathcal{L}_0^{n,\bar u(r),\bar m(r)} f(r,\bar Y(r)) + \frac{\partial}{\partial r} f(r,\bar Y(r))\right) dr \tag{4.15}$$

are $(\bar{\mathcal{F}}_s)$-martingales for all bounded f having a continuous time derivative.

ii. *Fix $x \in G_n^*$. Let a measurable function $m : [t,T] \times G_n^* \to M_b$ and a strategy $\alpha \in A$ be given. Then there exists a filtered probability space $(\bar\Omega, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_s\}_{[t,T]}, \bar{\mathbb{P}}^n)$, and $(\bar{\mathcal{F}}_s)$-adapted RCLL processes $\bar X$, $\bar Y$, $\bar m$, and $\bar u$, taking values in G_n^*, $n^{-1}\mathbb{Z}^I$, M, and U, respectively, such that $\bar{\mathbb{P}}^n$-a.s., $\bar u = \alpha[\bar m]$, $\bar m(s) = m(s, \bar X(s)), s \in [t,T], \bar X = \Gamma^n[\bar Y], \bar X(t) = \bar Y(t) = x$, and the process defined as in (4.14), replacing $\mathcal{L}$ by $\tilde{\mathcal{L}}$, as well as the process defined as in (4.15), are $(\bar{\mathcal{F}}_s)$-martingales for all bounded f having a continuous time derivative.*

Proof of Proposition 3.2. We first prove the second assertion, namely, (3.14). Fix $t_0 \in [0,T)$, x_0, and $G^n \ni x^n \to x_0$. It follows from Lemma 4.1 that to show

$$\liminf_{n\to\infty} Q^n(t_0, x^n) \ge V^+(t_0, x_0), \tag{4.16}$$

it suffices that for each $\beta \in B_b$ (where b is sufficiently large),

$$\liminf_{n\to\infty} Q^n(t_0, x^n) \ge c(t_0, x_0, \beta) := \inf_{u \in \bar U} c(t_0, x_0, u, \beta[u]). \tag{4.17}$$

We fix such β and turn to prove (4.17).

Since U is compact and convex, and the objective function in (4.6) is affine in u and concave in m, the outer minimum in (4.6) is achieved. We denote by $u = u^n(t,x)$ a minimizer in (4.6). Furthermore, the minimizer $u^n(t,x)$ can be selected as a measurable function of t and x (see Theorem 2.2 in [22]). Then from (4.6) we have

$$\mathcal{L}^{n,u^n(t,x),m} Q^n(t,x) + \frac{d}{dt} Q^n(t,x) + h(x) - \rho(u^n(t,x), m) \le 0, \qquad m \in M, t \in [t_0,T], x \in G^n. \tag{4.18}$$

We invoke Lemma 4.3(i) with $u = u^n$ and the given β. Denote by $\bar{\mathbb{E}}^n$ expectation w.r.t. the measure $\bar{\mathbb{P}}^n$ from that lemma. Replacing x by $\bar X_t^n$ in (4.18), $u^n(t,x)$ by $\bar u^n(t) = u^n(t, \bar X^n(t))$, and $m = \bar m^n(t) := \beta[\bar u^n](t)$, we have, $\bar{\mathbb{P}}^n$-a.s.,

$$\mathcal{L}^{n,\bar u^n(t),\bar m^n(t)} Q^n(t, \bar X^n(t)) + \frac{d}{dt} Q^n(t, \bar X^n(t)) + h(\bar X^n(t)) - \rho(\bar u^n(t), \bar m^n(t)) \le 0.$$

Consider the stopping times $\tau_L := \inf\{t \ge t_0 : \|\bar X_t^n - x_0\| \ge L\}$ for positive L. Let $B = B_{2L}(x_0)$ be the intersection of G^n and the I-dimensional ball of radius L, centered at x_0. Lemma 4.3 cannot be applied with $f(t,x) = Q^n(t,x)$ because this function may not be bounded. To apply the lemma we fix a function f that agrees with Q^n on B, is bounded, and continuously differentiable in t, which clearly exists due to the fact that Q^n is continuously differentiable and locally bounded. We substitute $s = T \wedge \tau_L$ and $f = Q^n$ in (4.14). Due to the local structure of the operator $\mathcal{L}^n$,

$\mathcal{L}^{n,\bar{u}^n(t),\bar{m}^n(t)} f(t, \bar{X}^n(t)) = \mathcal{L}^{n,\bar{u}^n(t),\bar{m}^n(t)} Q^n(t, \bar{X}^n(t))$ holds for $t \in [t_0, T \wedge \tau_L]$. Thus using inequality in the above display and taking expectation,

$$\bar{\mathbb{E}}^n \Bigg[Q^n(T \wedge \tau_L, \bar{X}^n(T \wedge \tau_L)) - Q^n(t_0, x^n) \\ + \int_{t_0}^{T\wedge\tau_L} (h(\bar{X}^n(r)) - \rho(\bar{u}^n(r), \bar{m}^n(r)))dr \Bigg] \le 0.$$

Denote by $\kappa_{n,L}$ the random variable inside the expectation. The assumed monotonicity of h and g implies that they are bounded below by $h(0)$ and $g(0)$, respectively. It follows by definition of Q^n that it is also bounded below. Since we also have $\bar{m}^n(r) \in M_b$, $\kappa_{n,L}$ is bounded below by a constant not depending on L. Hence, using Fatou's lemma, $\bar{\mathbb{E}}^n \liminf_L \kappa_{n,L} \le 0$. Since $\beta \in B_b$, the processes $\bar{X}^n$ are dominated in law by n^{-1} times a Poisson process with a given rate (that depends only on n). Hence $\lim_L \tau_L = \infty$, a.s., and

$$\bar{\mathbb{E}}^n \left[Q^n(T, \bar{X}^n(T)) - Q^n(t_0, x^n) + \int_{t_0}^{T} (h(\bar{X}^n(r)) - \rho(\bar{u}^n(r), \bar{m}^n(r)))dr \right] \le 0.$$

Since $Q^n(T, x) = g(x)$ for all x, we obtain

$$Q^n(t_0, x^n) \ge \bar{\mathbb{E}}^n \left[\int_{t_0}^{T} (h(\bar{X}^n(r)) - \rho(\bar{u}^n(r), \bar{m}^n(r)))dr + g(\bar{X}^n(T)) \right]. \tag{4.19}$$

From the definition of c we have $\bar{\mathbb{P}}^n$-a.s.,

$$\int_{t_0}^{T} [h(\varphi^n(r)) - \rho(\bar{u}^n(r), \bar{m}^n(r))]\, dr + g(\varphi^n(T)) \ge c(t_0, x_0, \beta), \tag{4.20}$$

where $\varphi^n = \Gamma(\psi^n)$ and $\psi^n(s) := x_0 + \int_{t_0}^{s} v(\bar{u}^n(r), \bar{m}^n(r))dr$, $s \in [t_0, T]$. Combining (4.19) and (4.20),

$$Q^n(t_0, x^n) \ge c(t_0, x_0, \beta) - \varepsilon_n,$$

where

$$\varepsilon_n = \bar{\mathbb{E}}^n \left[\int_{t_0}^{T} |h(\bar{X}^n(r)) - h(\varphi^n(r))|dr + |g(\bar{X}^n(T)) - g(\varphi^n(T))| \right]. \tag{4.21}$$

Using the Lipschitz continuity of h, g and the map Γ (in the sense of (3.10)), denoting $\|f\|^* := \sup_{[t_0,T]} \|f\|$, we have

$$\varepsilon_n \le c_1 \bar{\mathbb{E}}^n[\|\bar{Y}^n - \psi^n\|^*],$$

where c_1 is a constant not depending on n.

Toward proving that ε_n converges to zero, write $\bar{m}^n(s) = (\bar{\lambda}_i^n(s), \bar{\mu}_{ij}^n(s))$, and observe by (4.15) with $f(s, y) = y_i$, and (2.11), that

$$\begin{aligned} \bar{Y}^n(s) - x^n &= \int_{t_0}^{s} \left[\sum_i \bar{\lambda}_i^n(r) e_i - \sum_{ij} \bar{\mu}_{ij}^n(r) \bar{u}_{ij}^n(r) e_i \right] dr + \eta_1^n(s) \\ &= \int_{t_0}^{s} v(\bar{u}^n(r), \beta[\bar{u}^n](r))dr + \eta_1^n(s) \\ &= \psi^n(s) - x_0 + \eta_1^n(s), \end{aligned}$$

where each of the components of η_1^n is a zero mean martingale. Given i, write M^n for the ith component $\langle e_i, \eta_1^n\rangle$. By the Burkholder–Davis–Gundy inequality,

$$\bar{\mathbb{E}}^n\{(\|M\|^*)^2\} \le c_2\bar{\mathbb{E}}^n\{[M^n, M^n]_T\},$$

where c_2 is a universal constant, and $[M^n, M^n]$ is the quadratic variation process (see [23, pp. 58 and 175]). Note that M^n has sample paths that are piecewise absolutely continuous, null at zero. Hence $[M^n, M^n]_T$ is given by $\sum_{s\le T} \Delta M^n(s)^2$ (see, for example, [23, Theorem 22(ii), p. 59]). Each jump of M^n is of size n^{-1}. Hence $\bar{\mathbb{E}}^n\{(\|M^n\|^*)^2\} \le c_2 n^{-2}\bar{\mathbb{E}}^n[N^n]$, where N^n is the number of jumps of η_1^n (equivalently, $\bar{Y}^n$) in the interval. Since $\bar{m}^n$ is bounded, N^n is dominated by a Poisson r.v. of mean $O(n)$. This shows $\bar{\mathbb{E}}^n\{(\|M^n\|^*)^2\} \le O(n^{-1})$. As a consequence, $\bar{\mathbb{E}}^n\|\eta_1^n\|^* \to 0$, and since $x^n \to x_0$, $\varepsilon_n \to 0$. This shows (4.17) and completes the proof of the second assertion of the proposition.

We next prove the first assertion of the proposition. To this end, we fix t_0 and a sequence $G_n^* \ni x^n \to x_0$. To prove

$$\limsup_{n\to\infty} R^n(t_0, x^n) \le V^-(t_0, x_0), \tag{4.22}$$

it suffices to show that, for any $\alpha \in A$,

$$\limsup_{n\to\infty} R^n(t_0, x^n) \le \tilde{c}(t_0, x_0, \alpha) := \sup_{m\in\bar{M}} c(t_0, x_0, \alpha[m], m). \tag{4.23}$$

Thus, fixing α, we will prove (4.23).

Using the relation between R^n and Q^n given in (4.5), it follows from (4.6) that the function $R^n : [0, T] \times G_n^* \to \mathbb{R}$ (that is, the restriction of R^n to $[0, T] \times G_n^*$, that we still denote by R^n) is continuously differentiable in t for every x, and satisfies

$$\begin{cases} \inf_{u\in U}\sup_{m\in M}(\tilde{\mathcal{L}}^{n,u,m}R^n(t, x) + \frac{d}{dt}R^n(t, x) + h(x) - \rho(u, m)) = 0, \\ \qquad t \in [0, T],\ x \in G_n^*, \\ R^n(T, x) = g(x), \qquad x \in G_n^*. \end{cases} \tag{4.24}$$

Corollary 37.3.2 of [24] states that if C and D are closed, convex nonempty Euclidean sets, F is a continuous, finite concave-convex function on $C \times D$, and either C or D is bounded, then $\inf_{d\in D}\sup_{c\in C} F(c, d) = \sup_{c\in C}\inf_{d\in D} F(c, d)$. Notice that the objective function, $F(u, m) = \tilde{\mathcal{L}}^{n,u,m}R^n(t, x) - \rho(u, m)$ is continuous on $U \times M$, convex in u (in fact, affine), and concave in m. Hence

$$\inf_{u\in U}\sup_{m\in M} F(u, m) = \sup_{m\in M}\inf_{u\in U} F(u, m),$$

and we may interchange the order of infimum and supremum in (4.24). As the infimum of concave functions, the map

$$m \mapsto \inf_{u\in U} F(u, m)$$

is concave, and since it is clearly finite, it is continuous. Moreover, the function in the above display has compact superlevel sets, since $m \mapsto \rho(u, m)$ has compact sublevel sets for each u. Hence, the supremum over m is achieved. We denote by $m = m^n(t, x)$ a point where this maximum is achieved. Thus

$$\tilde{\mathcal{L}}^{n,u,m^n(t,x)}R^n(t, x) + \frac{d}{dt}R^n(t, x) + h(x) - \rho(u, m^n(t, x)) \ge 0, \qquad u \in U, t \in [t_0, T], x \in G_n^*. \tag{4.25}$$

Toward using Lemma 4.3(ii), let us argue that m^n is bounded. Indeed, by the structure (4.13) of $\tilde{\mathcal{L}}$ and the estimate (3.5) on R^n, the first term on the left-hand side of (4.25) is bounded by $C\|m^n\|$, where C is a constant and $\|m^n\| = \sup_{t,x} \|m^n(t,x)\|$. Since by (3.7) we also have that $\frac{d}{dt}R^n(t,x)$ is uniformly bounded, and h is bounded by assumption, this gives for every u, t, x the inequality

$$\rho(u, m^n(t,x)) \le C(1 + \|m^n\|)$$

for some constant C independent of u, t, x, n. By the form (2.12) of ρ, noting that l is superlinear and selecting u bounded away from zero, it follows that $\gamma(\|m^n\|) \le C(1 + \|m^n\|)$, where γ is some function satisfying $\gamma(r)/r \to \infty$ as $r \to \infty$. This shows that m^n are bounded.

Consider Lemma 4.3(ii) with $m = m^n$ and the given α. Replace x by $\bar{X}^n_t$ in (4.25) (note that $\bar{X}^n$ takes values in G^*_n), $m^n(t,x)$ by $\bar{m}^n(t) = m^n(t, \bar{X}^n(t))$, and u by $\bar{u}^n := \alpha[\bar{m}^n](t)$, to obtain, $\bar{\mathbb{P}}^n$-a.s.,

$$\tilde{\mathcal{L}}^{n,\bar{u}^n(t),\bar{m}^n(t)} R^n(t, \bar{X}^n(t)) + \frac{d}{dt} R^n(t, \bar{X}^n(t)) + h(\bar{X}^n(t)) - \rho(\bar{u}^n(t), \bar{m}^n(t)) \ge 0.$$

Take expectation in (4.14), substitute $s = T$, $f = R^n$, and use $R^n(T,x) = g(x)$ for all x, to obtain

$$R^n(t_0, x^n) \le \bar{\mathbb{E}}^n \left[\int_{t_0}^T (h(\bar{X}^n(s)) - \rho(u^n(s), \bar{m}^n(s)))ds + g(\bar{X}^n(T)) \right]. \tag{4.26}$$

Here, we have omitted an argument to go from a truncated version of R^n to R^n, analogous to that used in the first part of the proof. By definition of $\tilde{c}$ we have $\bar{\mathbb{P}}^n$-a.s.,

$$\int_{t_0}^T [h(\varphi^n(s)) - \rho(\bar{u}^n(s), \bar{m}^n(s))]\, ds + g(\varphi^n_T) \le \tilde{c}(t_0, x_0, \alpha), \tag{4.27}$$

where $\varphi^n = \Gamma(\psi^n)$ and $\psi^n(s) := x_0 + \int_t^s v(\bar{u}^n(r), \bar{m}^n(r))dr$, $s \in [t_0, T]$. Thus

$$R^n(t_0, x^n) \le \tilde{c}(t_0, x_0, \alpha) + \tilde{\varepsilon}_n,$$

where $\tilde{\varepsilon}_n$ has the same form as ε_n of (4.21). The argument that $\tilde{\varepsilon}_n \to 0$ is similar to that for ε_n. This establishes (4.23) and completes the proof of assertion (4.22). ∎

Remark 4.1. In the present set of admissible controls, the entries in the allocation matrix could be noninteger. We construct two models (a) and (b) accordingly so that those two constructions give lower and upper bounds of value, respectively. Now with a different set of admissible controls, one could follow the same approach by suitably choosing model (a) and model (b). For example, one might want to restrict to the case where the entries would be integer, we mean (1) a job cannot be split, (2) a server cannot split its attention. It turns out that for such integral allocation (IA) constraint, one may choose models (a) and (b) similar to the present ones along with an additional constraint of IA. Then it remains to check that Lemma 3.1, Proposition 3.2, and Lemma 4.2 are still correct. The proof of Lemma 4.2 needs the following additional observation. Minimization in (4.7) is of an affine function over a compact and convex set, so the infimum is achieved at the boundary, thus the IA constraint is satisfied by the minimizer. For Lemma 3.1 and Proposition 3.2 one can mimic the respective present proofs.

5. The PDE and the differential game. In this section we establish uniqueness of solutions to the PDE (2.14) by proving Proposition 3.1, and state and prove Lemma 5.1 regarding regularity of V^+ and V^-. We also give the proof of Proposition 3.3, regarding V^+ and V^- being solutions to the PDE.

Proof of Proposition 3.1. In this proof we write $H(p,x)$ for $H(p)+h(x)$. We will use the continuity of $p \mapsto H(p)$, that can be verified directly, using convexity of $H(p,u,m) = \langle p, v(u,m)\rangle - \rho(u,m)$ in u and concavity in m.

For $a > 0$ let

$$U(t,x) := u(t,x) - ae^{-\langle \mathbf{e},x\rangle},$$
$$V(t,x) := v(t,x) + ae^{-\langle \mathbf{e},x\rangle},$$

where $\mathbf{e} = \sum_{i=1}^K e_i \in \mathbb{R}^K$. To prove that $u \le v$ we argue by contradiction and assume that

$$\varrho := \sup_{[0,T]\times G} [u(t,x) - v(t,x)] > 0.$$

Hence there exists $(\tau, z) \in [0,T] \times G$ and $a_0 > 0$ such that for all $a \in (0, a_0)$,

$$U(\tau,z) - V(\tau,z) \ge \frac{2}{3}\varrho. \tag{5.1}$$

For $\varepsilon, \delta > 0$, introduce

$$\Phi(s,t,x,y) := U(s,x) - V(t,y) - \frac{1}{\varepsilon^2}\|x-y\|^2 - \frac{1}{\varepsilon^2}(t-s)^2 - \varepsilon(\|x\|^2 + \|y\|^2) - \delta(2T-s-t).$$

Note that $|U(s,x)| + |V(t,y)| \le c + c|x| + c|y|$, where $c = c(u,v,a)$. Thus $\Phi \downarrow -\infty$ as $(\|x\|^2 + \|y\|^2) \uparrow \infty$, and Φ admits a maximizer $(s^\varepsilon, t^\varepsilon, x^\varepsilon, y^\varepsilon)$ over $[0,T]^2 \times G^2$. Therefore there exist positive numbers ε and δ such that

$$\Phi(s^\varepsilon, t^\varepsilon, x^\varepsilon, y^\varepsilon) \;\ge\; \Phi(\tau,\tau,z,z) \;\ge\; \frac{2}{3}\varrho - 2\varepsilon\|z\|^2 - 2\delta(T-\tau) \;\ge\; \frac{\varrho}{2} \tag{5.2}$$

for all $a \in (0,a_0)$. In what follows, δ remains fixed while ε is made smaller (eventually, a will also be taken small). Since $\Phi(s^\varepsilon, t^\varepsilon, x^\varepsilon, y^\varepsilon) > 0$, we have

$$U(s^\varepsilon, x^\varepsilon) - V(t^\varepsilon, y^\varepsilon) \;>\; \frac{1}{\varepsilon^2}\|x^\varepsilon - y^\varepsilon\|^2 + \frac{1}{\varepsilon^2}(t^\varepsilon - s^\varepsilon)^2 + \varepsilon(\|x^\varepsilon\|^2 + \|y^\varepsilon\|^2) + \delta(2T - s^\varepsilon - t^\varepsilon). \tag{5.3}$$

Since u and v both satisfy the terminal condition, namely, $u(T,\cdot) = v(T,\cdot) = g(\cdot)$ and U, V, and g are Lipschitz, there are constants k_1 and k_2 such that

$$k_1\|x^\varepsilon - y^\varepsilon\| + k_2 \ge U(s^\varepsilon, x^\varepsilon) - V(t^\varepsilon, y^\varepsilon). \tag{5.4}$$

We argue that the left side is bounded for all positive ε. If not true, then there is a small enough ε such that $\frac{1}{\varepsilon^2}\|x^\varepsilon - y^\varepsilon\| \ge k_1\|x^\varepsilon - y^\varepsilon\| + k_2$. But this along with (5.4) and (5.3) leads to a contradiction. Thus the left side of (5.3) is bounded for all positive ε. Hence from (5.3) we conclude the following estimates:

$$\|x^\varepsilon - y^\varepsilon\| \le O(\varepsilon), \qquad |t^\varepsilon - s^\varepsilon| \le O(\varepsilon), \qquad \|x^\varepsilon\| \le O\left(\frac{1}{\sqrt{\varepsilon}}\right), \qquad \|y^\varepsilon\| \le O\left(\frac{1}{\sqrt{\varepsilon}}\right). \tag{5.5}$$

Next we show that

$$\|x^\varepsilon - y^\varepsilon\| + |t^\varepsilon - s^\varepsilon| = o(\varepsilon). \tag{5.6}$$

Using $\Phi(s^\varepsilon, t^\varepsilon, x^\varepsilon, y^\varepsilon) \ge \Phi(s^\varepsilon, s^\varepsilon, x^\varepsilon, x^\varepsilon)$, we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon^2}(\|x^\varepsilon - y^\varepsilon\|^2 + (t^\varepsilon - s^\varepsilon)^2) \\
&\quad \le V(s^\varepsilon, x^\varepsilon) - V(t^\varepsilon, y^\varepsilon) + \varepsilon(\|x^\varepsilon\|^2 - \|y^\varepsilon\|^2) + \delta(t^\varepsilon - s^\varepsilon) \\
&\quad \le \omega_V(\|x^\varepsilon - y^\varepsilon\| + |t^\varepsilon - s^\varepsilon|) + \varepsilon\langle x^\varepsilon + y^\varepsilon, x^\varepsilon - y^\varepsilon\rangle + \delta(t^\varepsilon - s^\varepsilon),
\end{aligned}$$

where ω_V is the modulus of continuity of V. Therefore (5.6) follows by using the estimates (5.5) in the above inequality. Next we show that

$$s^\varepsilon, t^\varepsilon < T \text{ for all sufficiently small } \varepsilon > 0. \tag{5.7}$$

To this end, note by (5.2) that

$$u(s^\varepsilon, x^\varepsilon) - v(t^\varepsilon, y^\varepsilon) \;\ge\; \frac{\varrho}{2}. \tag{5.8}$$

Now if any of s^ε and t^ε equals T, then $|T - s^\varepsilon| = o(\varepsilon)$ and $|T - t^\varepsilon| = o(\varepsilon)$ hold. Thus using Lipschitz continuity of g and denoting by ω_u and ω_v the modulus of continuity of u and v, respectively,

$$\begin{aligned}
|u(s^\varepsilon, x^\varepsilon) - v(t^\varepsilon, y^\varepsilon)| &\le |u(s^\varepsilon, x^\varepsilon) - g(x^\varepsilon)| + |v(t^\varepsilon, y^\varepsilon) - g(y^\varepsilon)| + |g(x^\varepsilon) - g(y^\varepsilon)| \\
&\le \omega_u(T - s^\varepsilon) + \omega_v(T - t^\varepsilon) + C_g\|x^\varepsilon - y^\varepsilon\| \to 0
\end{aligned}$$

by (5.6). This contradicts (5.8). Therefore (5.7) holds.

Let

$$\theta(s, x) := \frac{1}{\varepsilon^2}\|x - y^\varepsilon\|^2 + \frac{1}{\varepsilon^2}(t^\varepsilon - s)^2 + \varepsilon\|x\|^2 + \delta(T - s) + ae^{-\langle \mathbf{e}, x\rangle}.$$

By the definition of $(s^\varepsilon, x^\varepsilon)$, $(s, x) \mapsto u(s, x) - \theta(s, x)$ has local maximum at $(s^\varepsilon, x^\varepsilon)$. Since $D\theta(s^\varepsilon, x^\varepsilon) = \frac{2}{\varepsilon^2}(x^\varepsilon - y^\varepsilon) + 2\varepsilon x^\varepsilon - ae^{-\langle \mathbf{e}, x^\varepsilon\rangle}\mathbf{e}$ we have

$$\max_{i \in \mathbf{I}(x^\varepsilon)} \langle D\theta(s^\varepsilon, x^\varepsilon), e_i\rangle < 0. \tag{5.9}$$

Hence by definition of viscosity subsolution

$$\begin{aligned}
0 &\le \frac{\partial}{\partial s}\theta(s^\varepsilon, x^\varepsilon) + H(D\theta(s^\varepsilon, x^\varepsilon), x^\varepsilon) \\
&= -\frac{2}{\varepsilon^2}(t^\varepsilon - s^\varepsilon) - \delta + H(D\theta(s^\varepsilon, x^\varepsilon), x^\varepsilon).
\end{aligned}$$

Similarly, for the following test function

$$\vartheta(t, y) := -\left(\frac{1}{\varepsilon^2}\|x^\varepsilon - y\|^2 + \frac{1}{\varepsilon^2}(t - s^\varepsilon)^2 + \varepsilon\|y\|^2 + \delta(T - t)\right) - ae^{-\langle \mathbf{e}, y\rangle}$$

the map $(t, y) \mapsto v(t, y) - \vartheta(t, y)$ has local minimum at $(t^\varepsilon, y^\varepsilon)$. By an argument analogous to the one used for the earlier test function θ, we obtain an inequality like (5.9) and finally by the definition of viscosity supersolution we obtain

$$\begin{aligned}
0 &\ge \frac{\partial}{\partial s}\vartheta(t^\varepsilon, y^\varepsilon) + H(D\vartheta(t^\varepsilon, y^\varepsilon), y^\varepsilon) \\
&= -\frac{2}{\varepsilon^2}(t^\varepsilon - s^\varepsilon) + \delta + H(D\vartheta(t^\varepsilon, y^\varepsilon), y^\varepsilon).
\end{aligned}$$

From the above two inequalities we obtain

$$2\delta + H\left(D\vartheta(t^\varepsilon, y^\varepsilon), y^\varepsilon\right) - H\left(D\theta(s^\varepsilon, x^\varepsilon), x^\varepsilon\right) \le 0. \tag{5.10}$$

Again using (5.5) and (5.6),

$$\begin{aligned}
&\|D\vartheta(t^\varepsilon, y^\varepsilon) - D\theta(s^\varepsilon, x^\varepsilon)\| \\
&\quad = \left\| \frac{2}{\varepsilon^2}(x^\varepsilon - y^\varepsilon) - 2\varepsilon y^\varepsilon + ae^{-\langle \mathbf{e}, y^\varepsilon\rangle}\mathbf{e} - \left(\frac{2}{\varepsilon^2}(x^\varepsilon - y^\varepsilon) + 2\varepsilon x^\varepsilon - ae^{-\langle \mathbf{e}, x^\varepsilon\rangle}\mathbf{e}\right)\right\| \\
&\quad = \left\| -2\varepsilon(x^\varepsilon + y^\varepsilon) + a\left(e^{-\langle \mathbf{e}, x^\varepsilon\rangle} + e^{-\langle \mathbf{e}, y^\varepsilon\rangle}\right)\mathbf{e}\right\| \\
&\quad \le O(\sqrt{\varepsilon}) + 2Ia.
\end{aligned}$$

Therefore

$$\limsup_{\varepsilon\to 0} \|D\vartheta(t^\varepsilon, y^\varepsilon) - D\theta(s^\varepsilon, x^\varepsilon)\| \le 2Ia. \tag{5.11}$$

Recall that u and v are Lipschitz in x uniformly over $[0, T] \times G$, and denote the maximal of their Lipschitz constants by C. Let us argue that

$$\|D\theta(s^\varepsilon, x^\varepsilon)\| \le C, \qquad \|D\vartheta(t^\varepsilon, y^\varepsilon)\| \le C. \tag{5.12}$$

Because $(s, x) \mapsto u(s, x) - \theta(s, x)$ has a local maximum at $(s^\varepsilon, x^\varepsilon)$, we have for any $x \in G$,

$$\theta(s^\varepsilon, x) - \theta(s^\varepsilon, x^\varepsilon) \ge u(s^\varepsilon, x) - u(s^\varepsilon, x^\varepsilon) \ge -C\|x - x^\varepsilon\|,$$

and so $\|D\theta(s^\varepsilon, x^\varepsilon)\| \le C$ provided x^ε is an interior point. If $x^\varepsilon \in \partial G$ then from the above display we can still deduce $|\langle e_i, D\theta(s^\varepsilon, x^\varepsilon)\rangle| \le C$ for all $i \notin \mathbf{I}(x^\varepsilon)$. For $i \in \mathbf{I}(x^\varepsilon)$ we obtain $\langle D\theta(s^\varepsilon, x^\varepsilon), e_i\rangle \ge -C$ by the same inequality, which along with (5.9) again gives $|\langle D\theta(s^\varepsilon, x^\varepsilon), e_i\rangle| \le C$. This shows (5.12) holds for θ, and for ϑ the argument is similar.

Thus using the uniform continuity of $(p, x) \mapsto H(p, x)$ on $B_C \times G$ and denoting the corresponding modulus of continuity by ω_C, we obtain from (5.6), (5.10), and (5.11)

$$2\delta \le \omega_C(2Ia). \tag{5.13}$$

Note that the above holds for all $a \in (0, a_0)$ and choice of δ does not depend on a. This gives a contradiction for $\delta > 0$ fixed and small $a > 0$, and completes the proof of the result. ∎

LEMMA 5.1. *The functions $V^-, V^+ : [0, T] \times G \to \mathbb{R}$ are globally Lipschitz continuous.*

Proof. Fix $s, t \in [0, T]$ and $x, y \in G$. In view of (4.1), given a positive constant ε there exist $\beta^\varepsilon \in B_b$ such that

$$V^+(s, x) - \varepsilon \le \inf_{u\in\bar{U}} c(s, x, u, \beta^\varepsilon[u]).$$

For this particular β^ε, there exists a $u^\varepsilon \in \bar{U}$ such that

$$\begin{aligned}
c(t, y, u^\varepsilon, \beta^\varepsilon[u^\varepsilon]) &\le \inf_{\bar{U}} c(t, y, u, \beta^\varepsilon[u]) + \varepsilon \\
&\le \sup_{\beta\in B_b} \inf_{\bar{U}} c(t, y, u, \beta[u]) + \varepsilon \\
&= V^+(t, y) + \varepsilon.
\end{aligned}$$

Therefore

$$V^+(s,x) \le c(s,x,u^\varepsilon,\beta^\varepsilon[u^\varepsilon]) + \varepsilon, \qquad V^+(t,y) \ge c(t,y,u^\varepsilon,\beta^\varepsilon[u^\varepsilon]) - \varepsilon.$$

Thus

$$V^+(s,x) - V^+(t,y) \le c(s,x,u^\varepsilon,m^\varepsilon) - c(t,y,u^\varepsilon,m^\varepsilon) + 2\varepsilon, \tag{5.14}$$

where $m^\varepsilon := \beta^\varepsilon[u^\varepsilon]$. Note by the definition of $\bar{U}$ and $\bar{M}$ that u^ε and m^ε are defined over the interval $[0,T]$. Let $\varphi^i_\varepsilon := \Gamma(\psi^i_\varepsilon)$ for $i = 1,2$, where

$$\psi^1_\varepsilon(\tau) := x + \int_s^\tau v(u^\varepsilon(z), m^\varepsilon(z))dz, \qquad \tau \in [s,T],$$
$$\psi^2_\varepsilon(\tau) := y + \int_t^\tau v(u^\varepsilon(z), m^\varepsilon(z))dz, \qquad \tau \in [t,T].$$

If $s < t$, set $\psi^2_\varepsilon = y$ on $[s,t]$. Otherwise, set $\psi^1_\varepsilon = x$ on $[t,s]$. Note that ψ^1_ε and ψ^2_ε are Lipschitz continuous due to the upper bound of each component of m^ε. Note also that $\|\psi^1_\varepsilon - \psi^2_\varepsilon\|^* := \max_{[s\wedge t,T]} \|\psi^1_\varepsilon - \psi^2_\varepsilon\| \le \|x-y\| + C|t-s|$, for some constant C. Thus by (3.12) we have

$$\begin{aligned}
&c(s,x,u^\varepsilon,m^\varepsilon) - c(t,y,u^\varepsilon,m^\varepsilon)\\
&\quad = \int_t^T \left(h(\varphi^1_\varepsilon(z))dz - h(\varphi^2_\varepsilon(z))\right) dz\\
&\qquad + \int_s^t \left(h(\varphi^1_\varepsilon(z)) - \rho(u^\varepsilon(z), m^\varepsilon(z))\right) dz + g(\varphi^1_\varepsilon(T)) - g(\varphi^2_\varepsilon(T))\\
&\quad \le \int_t^T \Big(h \circ \Gamma(\psi^1_\varepsilon)(z) - h \circ \Gamma(\psi^2_\varepsilon)(z)\Big) dz\\
&\qquad + \int_{s\wedge t}^{s\vee t} [h \circ \Gamma(\psi^1_\varepsilon)(z) + \max_{U\times M_b} \rho(u,m)]dz + C_g\|\varphi^1_\varepsilon(T) - \varphi^2_\varepsilon(T)\|\\
&\quad \le C_1\|\psi^1_\varepsilon - \psi^2_\varepsilon\|^* + C_2|t-s|,
\end{aligned}$$

where C_1 and C_2 are constants (that may depend on T) and we used the Lipschitz continuity of g, h, and Γ, and the boundedness of h. Hence the Lipschitz continuity of V^+ follows from (5.14) and the above inequality. Analogously, V^- can be shown to be Lipschitz continuous. □

Proof of Proposition 3.3. From (3.12) it is clear that both of V^+ and V^- satisfy the terminal condition in (2.14). Given the Lipschitz property of the value functions, that is proved in Lemma 5.1, the proof that V^+ and V^- are both sub- and supersolutions of (2.14) is similar to that of Theorem 6 in [4]. Therefore we only provide here the proof of one of the four statements. In particular, we show that V^+ is a subsolution of (2.14).

Let θ be smooth and let $(s,y) \in [0,T) \times G$ be a local maximum of $V^+ - \theta$. We assume without loss of generality that $V^+(s,y) = \theta(s,y)$. We need to show

$$[\theta_t(s,y) + H(D\theta(s,y)) + h(y)] \vee \max_{i\in\mathbf{I}(y)} \langle D\theta(s,y), e_i\rangle \ge 0.$$

We shall assume the contrary and reach a contradiction. Thus there exists $a > 0$ such that

$$\theta_t(s,y) + \inf_{u\in U}\sup_{m\in M} [\langle D\theta(s,y), v(u,m)\rangle - \rho(u,m)] + h(y) < -a$$

$$\text{and} \qquad \langle D\theta(s,y), e_i\rangle < -a \qquad \text{for all } i \in \mathbf{I}(y).$$

Therefore there exists a $u_0 \in U$ such that for all m

$$\theta_t(s,y) + \langle D\theta(s,y), v(u_0,m)\rangle - \rho(u_0,m) + h(y) < -\frac{a}{2}.$$

For any strategy β if $\bar{u}_0(t) \equiv u_0$, for all $t \geq s$

$$\theta_t(s,y) + \langle D\theta(s,y), v(\bar{u}_0(t), \beta[\bar{u}_0](t))\rangle - \rho(\bar{u}_0(t), \beta[\bar{u}_0](t)) + h(y) < -\frac{a}{2}.$$

Consider the trajectory $\varphi : [s,T] \to G$ satisfying (3.11) with $\varphi(s) = y$, $u = \bar{u}_0$, and $m = \beta[\bar{u}_0]$. From continuity of $D\theta, \theta_t, h$, and φ, there is a $\delta > 0$ such that for $t \in [s, s+\delta]$

$$\theta_t(t,\varphi(t)) + \langle D\theta(t,\varphi(t)), v(\bar{u}_0(t), \beta[\bar{u}_0](t))\rangle - \rho(\bar{u}_0(t), \beta[\bar{u}_0](t)) + h(\varphi(t)) < -\frac{a}{4}$$

$$\text{and} \qquad \langle D\theta(t,\varphi(t)), e_i\rangle < 0 \qquad \text{for all } i \in \mathbf{I}(y).$$

Indeed we can select a sufficiently small $\delta > 0$ so that in addition to the above, there exist $a_i \geq 0$ such that for $t \in [s, s+\delta]$

$$\dot{\varphi}(t) = v(\bar{u}_0(t), \beta[\bar{u}_0](t)) + \sum_{i\in\mathbf{I}(y)} a_i e_i.$$

Hence for $t \in [s, s+\delta]$

$$\begin{aligned}\frac{d}{dt}\theta(t,\varphi(t)) &= \theta_t + \langle D\theta(t,\varphi(t)), \dot{\varphi}(t)\rangle \\ &= \theta_t + \langle D\theta(t,\varphi(t)), v(\bar{u}_0(t), \beta[\bar{u}_0](t))\rangle + \sum_{i\in\mathbf{I}(y)} a_i \langle D\theta(t,\varphi(t)), e_i\rangle \\ &< \rho(\bar{u}_0(t), \beta[\bar{u}_0](t)) - h(\varphi(t)) - \frac{a}{4}.\end{aligned}$$

Therefore for any $\beta \in B$, and $0 < \varepsilon < \delta$,

$$\theta(s+\varepsilon, \varphi(s+\varepsilon)) - \theta(s,y) < \int_s^{s+\varepsilon} \Big(\rho(\bar{u}_0(t), \beta[\bar{u}_0](t)) - h(\varphi(t))\Big)dt - \frac{a}{4}\varepsilon. \tag{5.15}$$

Again for a fixed $\varepsilon \in [0,\delta]$, there is a $\beta_0 \in B$ such that

$$\begin{aligned}V^+(s,y) &= \sup_{\beta\in B}\inf_{u\in\bar{U}} \left(\int_s^T \Big(h(\varphi(t)) - \rho(u(t), \beta[u](t))\Big)dt + g(\varphi(T))\right) \\ &= \sup_{\beta\in B}\inf_{u\in\bar{U}} \left(\int_s^{s+\varepsilon} \Big(h(\varphi(t)) - \rho(u(t), \beta[u](t))\Big)dt + V^+(s+\varepsilon, \varphi(s+\varepsilon))\right) \\ &\leq \inf_{u\in\bar{U}} \left(\int_s^{s+\varepsilon} \Big(h(\varphi(t)) - \rho(u(t), \beta_0[u](t))\Big)dt + V^+(s+\varepsilon, \varphi(s+\varepsilon))\right) + \frac{a}{8}\varepsilon \\ &\leq \int_s^{s+\varepsilon} \Big(h(\varphi(t)) - \rho(\bar{u}_0(t), \beta_0[\bar{u}_0](t))\Big)dt + V^+(s+\varepsilon, \varphi(s+\varepsilon)) + \frac{a}{8}\varepsilon, \end{aligned} \tag{5.16}$$

where the second equality is due to a standard dynamic programming argument. Now by substituting $\beta = \beta_0$ in (5.15) and comparing that with (5.16), we get

$$V^+(s+\varepsilon, \varphi(s+\varepsilon)) - \theta(s+\varepsilon, \varphi(s+\varepsilon)) > \frac{a}{8}\varepsilon > 0 = V^+(s,y) - \theta(s,y)$$

for all $\varepsilon \in [0,\delta]$, contradicting the fact that $(s,y) \in [0,T) \times G$ is a local maximum of $V^+ - \theta$. This proves that V^+ is a subsolution. □

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