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ON POSITIVE RECURRENCE OF CONSTRAINED DIFFUSION PROCESSES

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Let $G \subset \mathbb{R}^k$ be a convex polyhedral cone with vertex at the origin given as the intersection of half spaces $\{G_i, i = 1, \ldots, N\}$, where n_i and d_i denote the inward normal and direction of constraint associated with G_i, respectively. Stability properties of a class of diffusion processes, constrained to take values in G, are studied under the assumption that the Skorokhod problem defined by the data $\{(n_i, d_i), i = 1, \ldots, N\}$ is well posed and the Skorokhod map is Lipschitz continuous. Explicit conditions on the drift coefficient, $b(\cdot)$, of the diffusion process are given under which the constrained process is positive recurrent and has a unique invariant measure. Define

$$\mathscr{C} \doteq \left\{ -\sum_{i=1}^{N} \alpha_i d_i ; \alpha_i \geq 0, i \in \{1, \ldots, N\} \right\}.$$

Then the key condition for stability is that there exists $\delta \in (0, \infty)$ and a bounded subset A of G such that for all $x \in G \backslash A$, $b(x) \in \mathscr{C}$ and $\operatorname{dist}(b(x), \partial \mathscr{C}) \geq \delta$, where $\partial \mathscr{C}$ denotes the boundary of $\mathscr{C}$.

1. Introduction. The stability properties of constrained stochastic processes are of central importance in the study of queuing systems that arise in computer networks, communications and manufacturing problems. In recent years there has been a significant progress in the study of stability of such systems [12, 13, 9, 3, 4, 5, 2, 15, 16, 1]. All the papers in the list above which treat the heavy traffic diffusion model consider the case where both the drift and the diffusion coefficients are constant. However, in many applications a homogeneous model of this kind is not well suited, and it is enough if we mention systems where there is a control that depends on the system's state. Although a similar motivation leads one to also study variable constraint directions on the boundary, we confine ourselves here to fixed directions. In fact, some basic stability properties of the corresponding Skorokhod map, that we take advantage of here, are not yet well understood in the setting of variable directions of constraint.

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In this paper we consider the stability properties of constrained diffusion processes when both the drift and the diffusion coefficients may be state dependent. Let $G \subset \mathbb{R}^k$ be a convex polyhedral cone with vertex at the origin given as the intersection of half spaces $\{G_i, i = 1, \ldots, N\}$, where n_i and d_i denote the inward normal and direction of constraint associated with G_i respectively. The stochastic processes considered in this paper will be constrained to take values in G. One of our central assumptions is that the Skorokhod problem defined by the data $\{(d_i, n_i); i = 1, \ldots, N\}$ is well posed on all of $D_G([0,\infty) : \mathbb{R}^k)$ (the space of functions ϕ which are right continuous, have left limits and $\phi(0) \in G$) and the Skorokhod map $\Gamma : D_G([0,\infty) : \mathbb{R}^k) \to D_G([0,\infty) : \mathbb{R}^k)$ is Lipschitz continuous. We refer the reader to [11, 6, 7] for sufficient conditions under which the Skorokhod map is Lipschitz continuous. The paper [19] studies networks of single class queues for which the Skorokhod problem associated to a diffusion approximation is regular. Some examples of feedforward networks which lead to a regular Skorokhod problem have been studied in [17, 18]. An example of a multiclass networks with feedback which leads to a regular Skorokhod problem has recently been studied in [8].

In this paper we consider the constrained diffusion process $\{X^x(t)\}_{t\geq 0}$ given as the unique solution of the equation

$$X^x(t) = \Gamma\left(x + \int_0^\cdot \sigma(X^x(s))dW(s) + \int_0^\cdot b(X^x(s))ds\right)(t), \qquad t \geq 0 \tag{1.1}$$

We assume global Lipschitz conditions on σ and b (cf. (2.2)), the boundedness (2.3) and uniform non degeneracy (Condition 2.4) of the diffusion coefficient σ. The main result we present is that under the above conditions the Markov process $\{X^x(t)\}_{t\geq 0}$ is positive recurrent and has a unique invariant measure if there exists a $\delta \in (0,\infty)$ and a bounded subset A of G such that for all $x \in G\backslash A$, $b(x) \in \mathscr{C}(\delta)$, where

$$\mathscr{C}(\delta) \doteq \{v \in \mathscr{C} : \operatorname{dist}(v, \partial C) \geq \delta\}$$

and

$$\mathscr{C} \doteq \left\{-\sum_{i=1}^N \alpha_i d_i; \alpha_i \geq 0, i \in \{1,\ldots,N\}\right\}$$

and $\partial\mathscr{C}$ denotes the boundary of $\mathscr{C}$. The Lipschitz conditions on σ and b are assumed to assure that there is a unique solution to the constrained diffusion process (1.1). As we point out in Remark 4.1, our main result continues to hold if these Lipschitz and growth assumptions are replaced by the assumption that (1.1) has a unique weak solution for every $x \in G$ and the solution is a Feller Markov process. It should also be observed that the non-degeneracy assumption on σ and the Feller property are used only in Section 4 in proving the ergodicity of the constrained diffusion process, and are not needed to prove stability.

As in [9, 15, 4] the key idea in the proof of this result is to study stability properties of a related deterministic dynamical system. In Theorem 3.2 we show that for each $\delta > 0$, the family of deterministic constrained trajectories defined as

$$z(t) \doteq \Gamma\left(x + \int_0^{\cdot} v(s)ds\right)(t), \tag{1.2}$$

for which $v(t) \in \mathscr{C}(\delta)$, $t \in [0, \infty)$, enjoys strong uniform stability properties if for some $\delta \in (0, \infty)$, $v(t) \in \mathscr{C}(\delta)$ for all $t \in [0, \infty)$. These stability properties enable us to use $T(x)$, the hitting time to the origin (cf. (3.9), as a Lyapunov function for the stability analysis of the stochastic problem. We study some basic properties of $T(\cdot)$ in Lemma 3.1. The key consequence of the stability of (1.2) is Lemma 3.1(iii). This result along with the Lipschitz property of the Skorokhod map leads to Lemma 4.1 which is the crucial step in relating the stability of the deterministic dynamical system (1.2) with that of the stochastic system (1.1). Stability and instability results for reflecting Brownian motion have been obtained in a number of different settings. The papers [21, 14] consider the two dimensional case with constant and non-constant directions of constraint on the two faces of the domain, respectively. In [13] conditions are presented which guarantee the stability of a multi-dimensional reflecting Brownian motion, and in addition characterizes conditions under which the invariant distribution has a product form distribution. The paper [2] also obtains sufficient conditions for stability for this class of processes.

Our main result is Theorem 4.1 where it is shown that there is a compact set $B \in G$ for which the hitting time:

$$\tau_B(x) \doteq \inf\{t : X^x(t) \in B\}$$

has finite expectation which as a function of x is bounded on compact subsets of G. Proofs of such results generally use a Lyapunov function that is in the domain of the generator of the process (e.g., twice continuously differentiable in our case). An interesting feature of the approach we use is that far less regularity is required of the Lyapunov function. The paper concludes with the proof of the positive recurrence and the uniqueness of the invariant measure for $\{X^x(t)\}$, which is standard due to the uniform non-degeneracy of the diffusion coefficient.

2. Definitions and formulation. Let $G \subset \mathbb{R}^k$ be the convex polyhedral cone in $\mathbb{R}^k$ with the vertex at origin given as the intersection of half spaces G_i, $i = 1, \ldots, N$. Let n_i be the unit vector associated with G_i via the relation

$$G_i = \{x \in \mathbb{R}^k : \langle x, n_i \rangle \geq 0\}.$$

Denote the boundary of a set $S \subset \mathbb{R}^k$ by ∂S. We will denote the set $\{x \in \partial G : \langle x, n_i \rangle = 0\}$ by F_i. For $x \in \partial G$, define the set, $n(x)$, of unit inward normals to G at x by

$$n(x) \doteq \{r : |r| = 1,\ \langle r, x - y \rangle \leq 0,\ \forall y \in G\}.$$

With each face F_i we associate a unit vector d_i such that $\langle d_i, n_i \rangle > 0$. This vector defines the *direction of constraint* associated with the face F_i. For $x \in$

∂G define

$$d(x) \doteq \left\{ d \in \mathbb{R}^k : d = \sum_{i \in \mathrm{In}(x)} \alpha_i d_i; \alpha_i \geq 0; \ |d| = 1 \right\},$$

where

$$\mathrm{In}(x) \doteq \{i \in \{1, 2, \ldots N\} : \langle x, n_i \rangle = 0\}.$$

Let $D([0,\infty) : \mathbb{R}^k)$ denote the set of functions mapping $[0,\infty)$ to $\mathbb{R}^k$ that are right continuous and have limits from the left. We endow $D([0,\infty) : \mathbb{R}^k)$ with the usual Skorokhod topology. Let

$$D_G([0,\infty) : \mathbb{R}^k) \doteq \{\psi \in D([0,\infty) : \mathbb{R}^k) : \psi(0) \in G\}.$$

For $\eta \in D([0,\infty) : \mathbb{R}^k)$ let $|\eta|(T)$ denote the total variation of η on $[0, T]$ with respect to the Euclidean norm on $\mathbb{R}^k$.

DEFINITION 2.1. Let $\psi \in D_G([0,\infty) : \mathbb{R}^k)$ be given. Then $(\phi, \eta) \in D([0,\infty) : \mathbb{R}^k) \times D([0,\infty) : \mathbb{R}^k)$ solves the Skorokhod problem (SP) for ψ with respect to G and d if and only if $\phi(0) = \psi(0)$, and for all $t \in [0,\infty)$:

(i) $\phi(t) = \psi(t) + \eta(t)$;
(ii) $\phi(t) \in G$;
(iii) $|\eta|(t) < \infty$;
(iv) $|\eta|(t) = \int_{[0,t]} I_{\{\phi(s) \in \partial G\}} d|\eta|(s)$;
(v) There exists Borel measurable $\gamma : [0,\infty) \to \mathbb{R}^k$ such that $\gamma(t) \in d(\phi(t))$, $d|\eta|$-almost everywhere and

$$\eta(t) = \int_{[0,t]} \gamma(s) d|\eta|(s).$$

On the domain $D \subset D_G([0,\infty) : \mathbb{R}^k)$ on which there is a unique solution to the Skorokhod problem we define the Skorokhod map (SM) Γ as $\Gamma(\psi) \doteq \phi$, if $(\phi, \psi - \phi)$ is the unique solution of the Skorokhod problem posed by ψ. We will make the following assumption on the regularity of the Skorokhod map defined by the data $\{(d_i, n_i); i = 1, 2, \ldots N\}$.

CONDITION 2.1. *The Skorokhod map is well defined on all of $D_G([0,\infty) : \mathbb{R}^k)$, that is, $D = D_G([0,\infty) : \mathbb{R}^k)$ and the SM is Lipschitz continuous in the following sense. There exists a $K < \infty$ such that for all $\phi_1, \phi_2 \in D_G([0,\infty) : \mathbb{R}^k)$:*

We will assume without loss of generality that $K \geq 1$. We refer the reader to [6] (or alternatively see [7]) for sufficient conditions under which this regularity property holds.

We now introduce the constrained diffusion process that will be studied in this paper. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space on which is given a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual hypotheses. Let $(W(t), \mathscr{F}_t)$ be a n-dimensional standard Wiener process on the above probability space. We will study the constrained diffusion process given as a solution to equation (1.1), namely,

$$X^x(t) = \Gamma\left(x + \int_0^{\cdot} \sigma(X^x(s))dW(s) + \int_0^{\cdot} b(X^x(s))ds\right)(t), \tag{2.1}$$

where $\sigma : G \to \mathbb{R}^{k\times k}$ and $b : G \to \mathbb{R}^k$ are maps satisfying the following condition:

CONDITION 2.2. *There exists $\gamma \in (0, \infty)$ for which*

$$|\sigma(x) - \sigma(y)| + |b(x) - b(y)| \leq \gamma|x - y| \qquad \forall x, y \in G \tag{2.2}$$

and

$$|\sigma(x)| \leq \gamma \qquad \forall x \in G. \tag{2.3}$$

Using the regularity assumption on the Skorokhod map it can be shown (cf. [10, 6]) that there is a well defined process satisfying (2.1). In fact, the classical method of Picard iteration gives the following:

THEOREM 2.1. *For all $x \in G$ there exists a unique pair of continuous $\{\mathscr{F}_t\}$ adapted processes $(X^x(t), k(t))_{t\geq 0}$ and a progressively measurable process $(\gamma(t))_{t\geq 0}$ such that the following hold:*

(i) *$X^x(t) \in G$, for all $t \geq 0$, a.s.*
(ii) *For all $t \geq 0$,*

$$X^x(t) = x + \int_0^t \sigma(X^x(s))dW(s) + \int_0^t b(X^x(s))ds + k(t),$$

a.s.
(iii) *For all $T \in [0, \infty)$,*

$$|k|(T) < \infty \qquad a.s.$$

(iv)

$$|k|(t) = \int_0^t I_{\{X^x(s)\in\partial G\}} d|k|(s),$$

and $k(t) = \int_0^t \gamma(s)d|k|(s)$ with $\gamma(s) \in d(X^x(s))$ a.e. $[d|k|]$.

REMARK 2.1. The process $X^x(\cdot)$ is the unique continuous $\{\mathscr{F}_t\}$ adapted process which satisfies the equation

$$X^x(t) = \Gamma\left(x + \int_0^{\cdot} \sigma(X^x(s))dW(s) + \int_0^{\cdot} b(X^x(s))ds\right)(t),$$

for all t a.s. Also, $X^x(\cdot)$ is a Feller Markov process.

We now proceed to formulate our central result. Define

$$\mathscr{C} \doteq \left\{ -\sum_{i=1}^{N} \alpha_i d_i : \alpha_i \geq 0;\ i \in \{1, \ldots, N\} \right\}$$

The cone $\mathscr{C}$ was used to characterize stability of certain semimartingale reflecting Brownian motions in [1].

For $\delta \in (0, \infty)$, define

$$\mathscr{C}(\delta) \doteq \{v \in \mathscr{C} : \operatorname{dist}(v, \partial C) \geq \delta\}.$$

Our next assumption on the diffusion model stipulates the permissible velocity directions.

CONDITION 2.3. *There exist a $\delta \in (0, \infty)$ and a bounded set $A \subset G$ such that for all $x \in G \backslash A$, $b(x) \in \mathscr{C}(\delta)$.*

Finally we will make the following uniform nondegeneracy assumption on the diffusion coefficient.

CONDITION 2.4. *There exists $c \in (0, \infty)$ such that for all $x \in G$ and $\alpha \in \mathbb{R}^k$*

$$\alpha'(\sigma(x)\sigma'(x))\alpha \geq c\alpha'\alpha.$$

Here is the main theorem of this paper.

THEOREM 2.2. *Assume that Conditions* 2.1–2.4 *hold. Then the strong Markov process $\{X^x(\cdot); x \in G\}$ is positive recurrent and has a unique invariant probability measure.*

In the rest of the paper we will assume that Conditions 2.1, 2.2, 2.3 and 2.4 hold.

3. Stability of constrained ODEs.

Let $v : [0, \infty) \to \mathbb{R}^k$ be a measurable map such that

$$\int_0^t |v(s)| ds < \infty \qquad \text{for all } t \in [0, \infty). \tag{3.1}$$

Let $x \in G$. In this section we will study the stability properties of the trajectory $z : [0, \infty) \to \mathbb{R}^k$ defined as

$$z(t) \doteq \Gamma\left(x + \int_0^{\cdot} v(s) ds\right)(t), \qquad t \in [0, \infty). \tag{3.2}$$

It is useful to rewrite the above trajectory as a solution of an ordinary differential equation. In order to do so we introduce the following notion of discrete projections (cf. [6, 7]). Define $\pi : \mathbb{R}^k \to G$ as follows:

$$\pi(y) \doteq \Gamma(\psi_y)(1), \quad y \in \mathbb{R}^k,$$

where $\psi_y \in D([0,\infty);\mathbb{R}^k)$ is given as

$$\psi_y(t) = \begin{cases} 0, & t \in [0,1), \\ y, & t \in [1,\infty). \end{cases}$$

In other words, π is a projection that is consistent with the given Skorokhod problem, in that the constrained version of any piecewise constant trajectory ψ can be found by recursively applying π. We also define the *projection of the velocity* $v \in \mathbb{R}^k$ *at* $x \in G$ by

$$\pi(x, v) \doteq \lim_{\Delta \to 0} \frac{\pi(x + \Delta v) - x}{\Delta}.$$

For a proof of the fact that the above limit exists we refer the reader to [1] where various properties of the projection map are also studied. In particular we will use the following facts (for proofs see [1]):

1. For $x \in G$, $\alpha, \beta, \gamma \geq 0$ and $v \in \mathbb{R}^k$:

$$\pi(\beta x, \alpha v + \gamma x) = \alpha \pi(x, v) + \gamma x. \tag{3.3}$$

2. For $v \in \mathbb{R}^k$, we have that

$$\pi(v) = 0 \qquad \text{if and only if } v \in \mathscr{C}. \tag{3.4}$$

The following theorem represents the trajectory in (3.2) as a solution of an ordinary differential equation (cf. [6]).

THEOREM 3.1. *Let* $v : [0,\infty) \to \mathbb{R}^k$ *satisfy* (3.1). *Then for all* $x \in G$, $z(\cdot)$ *defined via* (3.2) *is the unique absolutely continuous function such that*

$$\dot{z}(t) = \pi(z(t), v(t)) \qquad a.e.\ t,\ z(0) = x.$$

In Theorem 3.2 below we present a basic stability property of the above dynamical system.

THEOREM 3.2. *Let* v *be as in Theorem* 3.1. *Assume that there exists a* $\delta \in (0,\infty)$ *such that*

$$v(t) \in \mathscr{C}(\delta) \qquad \text{for all } t \in [0,\infty).$$

Let $x \in G$ *and* $z(\cdot)$ *be defined via* (3.2). *Then*

$$|z(t)| \leq \frac{K^2|x|^2}{K|x| + \delta t} \qquad \forall t \in [0,\infty),$$

where K *is the finite constant in* (2.3).

PROOF. In order to specify the initial point of the trajectory we will write the trajectory defined by (3.2) as $z(x,\cdot)$. Define the trajectory $\tilde{z}(\cdot)$ as

$$\tilde{z}(t) \doteq \Gamma\left(\int_0^{\cdot} v(s)ds\right)(t).$$

Theorem 3.1 implies that $\tilde{z}(\cdot)$ is the unique solution of

$$\dot{\tilde{z}}(t) = \pi(\tilde{z}(t), v(t)) \qquad a.e.\ t;\ \tilde{z}(0) = 0. \tag{3.5}$$

However since $v(t) \in \mathscr{C}(\delta) \subset \mathscr{C}$, we have from (3.4) that $\pi(0, v(t)) = 0$ for all $t \geq 0$ and so the zero trajectory solves (3.5). By Theorem 3.1 this implies that $\tilde{z}(t) \equiv 0$. Thus

$$\begin{aligned}\sup_{0\leq t<\infty} |z(x,t)| &= \sup_{0\leq t<\infty} |z(x,t) - \tilde{z}(t)| \\ &= \sup_{0\leq t<\infty} \left|\Gamma\left(x + \int_0^{\cdot} v(s)ds\right)(t) - \Gamma\left(\int_0^{\cdot} v(s)ds\right)(t)\right| \\ &\leq K|x|.\end{aligned} \tag{3.6}$$

The above inequalities in particular show that the theorem is true when $x = 0$. Henceforth we assume that $x \neq 0$. Define

$$\gamma \doteq \frac{\delta}{K|x|}$$

and

$$\psi(t) \doteq (1 + \gamma t) z(x, t). \tag{3.7}$$

Note that from (3.3) it follows that

$$\begin{aligned}\dot{\psi}(t) &= \gamma z(x,t) + (1+\gamma t)\pi(z(x,t), v(t)) \\ &= \pi\left(z(x,t), (1+\gamma t)v(t) + \gamma z(x,t)\right) \\ &= \pi\left(\psi(t), (1+\gamma t)\left(v(t) + \frac{\gamma}{1+\gamma t} z(x,t)\right)\right).\end{aligned}$$

By Theorem 3.1 we now have that $\psi(t) = \Gamma(x + f(\cdot))(t)$ for all $t \in [0, \infty)$ where

$$f(t) \doteq \int_0^t \left((1+\gamma s)\left(v(s) + \frac{\gamma}{1+\gamma s} z(x,s)\right)\right) ds.$$

Note that from (3.7) it follows that

$$\left|\frac{\gamma}{1+\gamma t} z(x,t)\right| \leq K|x| \frac{\delta}{K|x|} = \delta.$$

Thus if $v \in \mathscr{C}(\delta)$, then $v + \frac{\gamma}{(1+\gamma t)} z(x,t) \in \mathscr{C}$. From this observation it follows that for all $t \in [0, \infty)$,

$$u(t) \doteq (1+\gamma t)\left(v(t) + \frac{\gamma}{1+\gamma t} z(x,t)\right) \in \mathscr{C}.$$

Define the trajectory

$$\tilde{\psi}(t) \doteq \Gamma(f(\cdot))(t), \qquad t \in [0, \infty).$$

Then $\tilde{\psi}(\cdot)$ solves the equation

$$\dot{\tilde{\psi}}(t) = \pi(\tilde{\psi}(t), u(t)), \qquad \tilde{\psi}(0) = 0. \tag{3.8}$$

Since for all $t \in (0, \infty)$, $u(t) \in \mathscr{C}$ we have that $\pi(0, u(t)) = 0$. Thus the function $x(t) = 0$ for all $t \in (0, \infty)$ is a solution of (3.8). Now by the uniqueness of the solution of (3.8) (Theorem 3.1) we have that $\tilde{\psi}(t) = 0$ for all $t \in (0, \infty)$. Thus

$$\begin{aligned} |\psi(t)| &= |\psi(t) - \tilde{\psi}(t)| \\ &\le |\Gamma(x + f(\cdot))(t) - \Gamma(f(\cdot))(t)| \\ &\le K|x|, \end{aligned}$$

for all $t \in (0, \infty)$. Finally, from (3.7),

$$|z(x, t)| \le \frac{K|x|}{1 + \gamma t} = \frac{K^2|x|^2}{K|x| + \delta t}. \qquad \square$$

For $x \in G$ and $\delta \in (0, \infty)$ let $\mathscr{A}(x, \delta)$ be the collection of all absolutely continuous functions $z : [0, \infty) \to \mathbb{R}^k$ defined via (3.2) for some $v : [0, \infty) \to \mathscr{C}(\delta)$ which satisfies (3.1). Henceforth we will fix such a δ and abbreviate $\mathscr{A}(x, \delta)$ by $\mathscr{A}(x)$.

For a fixed $x \in G$, we now define the "hitting time to the origin" function as follows:

$$T(x) \doteq \sup_{z \in \mathscr{A}(x)} \inf\{t \in [0, \infty) : z(t) = 0\}. \tag{3.9}$$

We next study some of the properties of $T(x)$.

LEMMA 3.1. *There exist constants $c, C \in (0, \infty)$ depending only on K and δ such that the following hold*:

(i) *For all $x, y \in G$,*

$$|T(x) - T(y)| \le C|x - y|.$$

(ii) *$T(x) \ge c|x|$. Thus, in particular, for all $M \in (0, \infty)$ the set $\{x \in G : T(x) \le M\}$ is compact.*

(iii) *Fix $x \in G$ and let $z \in \mathscr{A}(x)$. Then for all $t > 0$,*

$$T(z(t)) \le (T(x) - t)^+.$$

PROOF. We first show that for all $x \in G$,

$$T(x) \le \frac{4K^2}{\delta}|x|. \tag{3.10}$$

Fix $x \in G$ and let $z \in \mathscr{A}(x)$ be arbitrary. From Theorem 3.2 we have that for all $t \in (0, \infty)$

$$|z(t)| \le \frac{K^2|x|^2}{K|x| + \delta t}.$$

Hence for all $t \geq T_1 \doteq 2K^2\delta^{-1}|x|$ one has $|z(t)| \leq |x|/2$. In general, if

$$T_n \doteq T_1 \sum_{k=0}^{n-1} 2^{-k}$$

then for $t \geq T_n$ one has that

$$|z(t)| \leq \frac{|x|}{2^n}.$$

Thus $z(t) = 0$ for all $t \geq 4K^2\delta^{-1}|x|$. Since $z \in \mathscr{A}(x)$ is arbitrary, (3.10) follows.

Now let $x, y \in G$ be arbitrary. Let $\{z_n\} \subset \mathscr{A}(x)$ be a sequence such that if

$$\tau_n \doteq \inf\{t : z_n(t) = 0\},$$

then

$$\tau_n \to T(x) \qquad \text{as } n \to \infty. \tag{3.11}$$

Note that z_n is given as

$$z_n(t) = \Gamma\left(x + \int_0^{\cdot} v_n(s)ds\right)(t), \qquad t \in [0, \infty)$$

for some v_n satisfying (3.2). Define for $t \in [0, \infty)$

$$w_n(t) \doteq \Gamma\left(y + \int_0^{\cdot} v_n(s)ds\right)(t).$$

From the Lipschitz property of Γ (Condition 2.1) we have that

$$\sup_{0 \leq t < \infty} |z_n(t) - w_n(t)| \leq K|x - y|.$$

Also clearly $w_n \in \mathscr{A}(y)$. Now let

$$\tau_n' \doteq \inf\{t \in (0, \infty) : w_n(t) = 0\}.$$

Fix n and suppose that $\tau_n \leq \tau_n'$. Then

$$|w_n(\tau_n)| = |w_n(\tau_n) - z_n(\tau_n)| \leq K|x - y|.$$

Hence from (3.10), letting $C \doteq 4K^3\delta^{-1}$,

$$\tau_n' \leq \tau_n + C|x - y|.$$

Similarly it can be seen that if $\tau_n' \leq \tau_n$, then

$$\tau_n \leq \tau_n' + C|x - y|$$

and thus

$$|\tau_n - \tau_n'| \leq C|x - y|.$$

We therefore have that

$$\tau_n \leq \tau_n' + C|x - y| \leq T(y) + C|x - y|.$$

Sending $n \to \infty$, it follows from (3.11) that

$$T(x) \le T(y) + C|x-y|.$$

Since the role of x and y can be reversed, we have that

$$|T(x) - T(y)| \le C|x-y|,$$

and since x and y are arbitrary we have (i).

Next we show (ii). Fix some $v \in \mathscr{C}(\delta)$, and let $x \in G\backslash\{0\}$ be given. With $\iota : [0,\infty) \to [0,\infty)$ denoting the identity map, clearly the trajectory $\{\Gamma(x + v\iota)(t)\}_{t\ge 0}$ belongs to $\mathscr{A}(x)$. Note that

$$\begin{aligned}\sup_{0\le t\le M} |\Gamma(x+v\iota)(t) - x| &= \sup_{0\le t\le M} |\Gamma(x+v\iota)(t) - \Gamma(x+0\iota)(t)| \\ &\le KM|v|.\end{aligned}$$

Therefore, for any $M < |x|/K|v|$,

$$\inf_{0\le t\le M} |\Gamma(x+v\iota)(t)| \ge |x| - KM|v| > 0,$$

which implies that $T(x) > M$. Taking the supremum over $M < |x|/K|v|$ gives

$$T(x) \ge \frac{|x|}{K|v|}.$$

This proves (ii) with $c = 1/K|v|$.

Finally we prove (iii). Let $t > 0$ be fixed. If $T(z(s)) = 0$ for some $s \in [0,t]$ then the result is obviously true. Now suppose that $T(z(s)) > 0$ for all $s \in [0,t]$. Let $\beta > 0$ be arbitrary and $u \in \mathscr{A}(z(t))$ be such that $\tau \doteq \inf\{s \in [0,\infty) : u(s) = 0\}$ satisfies $\tau > T(z(t)) - \beta$. Define $\tilde{z} : [0,\infty) \to \mathbb{R}^k$ by

$$\tilde{z}(s) = \begin{cases} z(s), & s \le t, \\ u(s-t), & s > t. \end{cases}$$

Then $\tilde{z} \in \mathscr{A}(x)$ and

$$\begin{aligned} T(x) &\ge \inf\{s \in [0,\infty) : \tilde{z}(s) = 0\} \\ &= t + \tau \\ &\ge T(z(t)) + t - \beta. \end{aligned}$$

Since $\beta > 0$ is arbitrary we have that $T(z(t)) \le T(x) - t$. This proves the lemma. □

4. Stability of constrained diffusion processes.

We begin with the following lemma. For $x \in G$, let $\Omega_0(x)$ be a $P-$null set such that for all $\omega \notin \Omega_0(x)$ and $0 \le u < t < \infty$, $X^x(\cdot) = X^x(\cdot,\omega)$ satisfies

$$X^x(t) = \Gamma\left(X^x(u) + \int_0^{\cdot} b(X^x(u+s))ds + \int_0^{\cdot} \sigma(X^x(u+s))dW_u(s)\right)(t-u),$$

where $W_u(s) \doteq W(s+u)$.

LEMMA 4.1. *Let T be the function defined in* (3.9). *Fix $x \in G \setminus A$ and let $\{X^x(t)\}_{t\geq 0}$ be as in Theorem* 2.2. *Let $\Delta > 0$ and $u > 0$ be arbitrary. Fix $\omega \notin \Omega_0(x)$. Suppose that $X^x(t,\omega) \in G \setminus A$ for all $t \in (u, u+\Delta]$. Then*

$$T(X^x(u+\Delta,\omega)) \leq (T(X^x(u,\omega)) - \Delta)^+ + KC\overline{\nu}(\omega),$$

where C is as in Lemma 3.1(i) *and*

$$\overline{\nu} \doteq \sup_{u \leq s \leq u+\Delta} \left| \int_u^s \sigma(X^x(s))\,dW(s) \right|. \tag{4.1}$$

PROOF. In the proof we will suppress ω from the notation. We begin by noticing that for $t \in [u, u+\Delta)$, $X^x(t) = \tilde{X}(t-u)$, where

$$\begin{aligned} \tilde{X}(t) \doteq \Gamma\bigg(X^x(u) &+ \int_0^{\cdot} b(X^x(s+u))\,ds \\ &+ \int_0^{\cdot} \sigma(X^x(s+u))\,dW_u(s) \bigg)(t), \qquad 0 \leq t \leq \Delta. \end{aligned}$$

Now define a sequence of $\mathbb{R}^k$ valued stochastic processes $\{\tilde{Y}(t)\}_{0\leq t\leq\Delta}$ as follows.

$$\tilde{Y}(t) \doteq \Gamma\left(X^x(u) + \int_0^{\cdot} b(X^x(s+u))\,ds \right)(t). \tag{4.2}$$

Note that $\tilde{Y}(t)$ has absolutely continuous paths P-a.s., and that $b(X^x(s+u)) \in \mathscr{C}(\delta)$ for all $s \in [0,\Delta]$. Also, note that by Condition 2.1 we have

$$\begin{aligned} \sup\nolimits_{0\leq t\leq\Delta} |\tilde{X}(t) - \tilde{Y}(t)| &\leq K \sup\nolimits_{0\leq t\leq\Delta} \left| \int_0^t \sigma(X^x(s+u))\,dW_u(s) \right| \\ &= K\overline{\nu}. \end{aligned} \tag{4.3}$$

Using the Lipschitz property of T [Lemma 3.1 (i)] we have that

$$\begin{aligned} T(X^x(u+\Delta)) &= T(\tilde{X}(\Delta)) \\ &\leq T(\tilde{Y}(\Delta)) + KC\overline{\nu} \\ &\leq (T(X^x(u)) - \Delta)^+ + KC\overline{\nu}, \end{aligned}$$

where the last inequality follows from Lemma 3.1(iii). □

LEMMA 4.2. *Suppose that $\{\alpha_i(t)\}; i = 1, 2, \ldots, l$ are $\mathbb{R}^k$ valued $\sigma\{W(s) : 0 \leq s \leq t\}$-progressively measurable processes such that there exists $\overline{\alpha} \in (0,\infty)$ for which*

$$|\alpha_i(t)| \leq \overline{\alpha},$$

for all $t \in (0,\infty)$, $i \in \{1,\ldots,l\}$, P-a.s. Then for $\lambda \in (0,\infty)$

$$\mathbb{E}\left(\exp\left\{ \lambda \sum_{i=1}^{l} \left| \int_0^t \langle \alpha_i(s), dW(s) \rangle \right| \right\} \right) \leq 2\exp\left\{ \frac{l^2\lambda^2\overline{\alpha}^2 t}{2} \right\},$$

where $\langle \cdot,\cdot \rangle$ is the usual inner product in $\mathbb{R}^k$.

PROOF. We first consider the case when $l = 1$. Observe that

$$E\left(\exp\left(\lambda \int_0^t \langle \alpha_1(s), dW(s)\rangle - \tfrac{1}{2}\lambda^2 \int_0^t |\alpha_1(s)|^2 ds\right)\right) = 1$$

and

$$E\left(\exp\left(-\lambda \int_0^t \langle \alpha_1(s), dW(s)\rangle - \tfrac{1}{2}\lambda^2 \int_0^t |\alpha_1(s)|^2 ds\right)\right) = 1.$$

Using the upper bound on $|\alpha_1(\cdot)|$ we now have that

$$\begin{aligned}
E\left(\exp\left(\lambda \left|\int_0^t \langle \alpha_1(s), dW(s)\rangle\right|\right)\right) &\le E\left(\exp\left(\lambda \int_0^t \langle \alpha_1(s), dW(s)\rangle\right)\right)\\
&\quad + E\left(\exp\left(-\lambda \int_0^t \langle \alpha_1(s), dW(s)\rangle\right)\right)\\
&\le \exp\left(\frac{\lambda^2\overline{\alpha}^2 t}{2}\right) + \exp\left(\frac{\lambda^2\overline{\alpha}^2 t}{2}\right)\\
&= 2\exp\left(\frac{\lambda^2\overline{\alpha}^2 t}{2}\right).
\end{aligned}$$

This proves the lemma for the case $l = 1$. Now we consider the case $l > 1$. Note that

$$\begin{aligned}
&\mathbb{E}\left(\exp\left\{\lambda \sum_{i=1}^{l} \left|\int_0^t \langle \alpha_i(s), dW(s)\rangle\right|\right\}\right)\\
&\quad \le \mathbb{E}\left(\prod_{i=1}^{l} \exp\left\{\lambda \left|\int_0^t \langle \alpha_i(s), dW(s)\rangle\right|\right\}\right)\\
&\quad \le \left(\prod_{i=1}^{l} \mathbb{E}\left(\exp\left\{l\lambda \left|\int_0^t \langle \alpha_i(s), dW(s)\rangle\right|\right\}\right)\right)^{\frac{1}{l}}\\
&\quad \le 2\left(\exp\left\{\frac{l^3\lambda^2\overline{\alpha}^2 t}{2}\right\}\right)^{\frac{1}{l}}\\
&\quad = 2\exp\left(\frac{l^2\lambda^2\overline{\alpha}^2 t}{2}\right).
\end{aligned}$$

□

In what follows, we will denote the set of positive integers by $\mathbb{N}$.

LEMMA 4.3. *Let $x \in G$ and $\Delta > 0$ be fixed. For $n \in \mathbb{N}$ let ν_n be defined as follows*:

$$\nu_n \doteq \sup_{(n-1)\Delta \le s \le n\Delta} \left|\int_{(n-1)\Delta}^{s} \sigma(X^x(s))dW(s)\right|. \tag{4.4}$$

Then for any $\kappa \in (0,\infty)$ and $m, n \in \mathbb{N}$; $m \le n$,

$$\mathbb{E}\left(\exp\left\{\kappa \sum_{i=m}^{n} \nu_i\right\}\right) \le \left[2\sqrt{2}\exp\left(k^2\kappa^2\gamma^2\Delta\right)\right]^{(n-m+1)},$$

where γ is as in Condition 2.2.

PROOF. For $t > 0$, let

$$\mathscr{G}_t \doteq \sigma\{W(s) : 0 \le s \le t\}.$$

Then

$$\mathbb{E}\exp\left(\kappa\sum_{i=m}^{n}\nu_i\right) = \mathbb{E}\left(\exp\left\{\kappa\sum_{i=m}^{n-1}\nu_i\right\}\left(\mathbb{E}\left(\exp\{\kappa\nu_n\} \mid \mathscr{G}_{(n-1)\Delta}\right)\right)\right). \tag{4.5}$$

Now

$$\begin{aligned}
&\mathbb{E}\left(\exp\{\kappa\nu_n\} \mid \mathscr{G}_{(n-1)\Delta}\right)\\
&\quad = \mathbb{E}\left(\sup_{(n-1)\Delta\le s\le n\Delta}\exp\left(\kappa\left|\int_{(n-1)\Delta}^{s}\sigma(X^x(u))\,dW(u)\right|\right)\,\middle|\,\mathscr{G}_{(n-1)\Delta}\right)\\
&\quad = \mathbb{E}\left(\sup_{(n-1)\Delta\le s\le n\Delta}\exp\left(\kappa\left|\int_{(n-1)\Delta}^{s}\sigma(X^x(u))\,dW(u)\right|\right)\,\middle|\,X^x((n-1)\Delta)\right),
\end{aligned}$$

where the last step follows from the Markov property of X^x.

An application of Doob's maximal inequality for submartingales yields that the last expression is bounded above by

$$2\left(\mathbb{E}\left(\exp\left\{2\kappa\left|\int_{(n-1)\Delta}^{n\Delta}\sigma(X(u))\,dW(u)\right|\right\}\,\middle|\,X^x((n-1)\Delta)\right)\right)^{\frac{1}{2}}.$$

By an application of Lemma 4.2 and the observation that for positive real numbers $x_1,\dots x_k$, $\sqrt{\sum_{i=1}^{k}x_i^2} \le \sum_{i=1}^{k}x_i$ we have that the last expression is bounded above by

$$2\sqrt{2}e^{k^2\kappa^2\gamma^2\Delta}.$$

Using this observation in (4.5) we have the result by iterating. □

For $\Delta > 0$, let

$$B^{\Delta} \doteq \{y \in G : T(y) \le \Delta\}.$$

Let $\{X^x(t)\}_{t\ge 0}$ be as in Theorem 2.1. Given a compact set $B \subset G$, let

$$\tau_B(x) \doteq \inf\{t : X^x(t) \in B\}. \tag{4.6}$$

THEOREM 4.1. *Let $\{X^x(t)\}_{t\geq 0}$ be as in Theorem 2.2. Then there exists $\Delta \in (0,\infty)$ such that for all $M \in (0,\infty)$,*

$$\sup_{x:|x|\leq M} \mathbb{E}(\tau_{B^\Delta}(x)) < \infty.$$

PROOF. Without loss of generality we can assume that Δ is chosen large enough so that $B^\Delta \supset A$.

Let $\Delta \in (0,\infty)$ and let

$$A_n \doteq \left\{\omega : \inf_{s\in[0,n\Delta]} T(X^x(s)) > \Delta\right\}.$$

Then

$$P(A_n) \leq P\left(\Delta < T(X^x(n\Delta)) \leq T(x) - n\Delta + CK\sum_{j=1}^{n} \nu_j\right), \tag{4.7}$$

where $\{\nu_j\}_{j=1}^n$ are as in (4.4) and the inequality follows from Lemma 4.1. Next observe that the probability on the right side of (4.7) is bounded above by

$$\begin{aligned}
&P\left(CK\sum_{i=1}^{n}\nu_i \geq (n+1)\Delta - T(x)\right)\\
&\quad\leq \frac{\mathbb{E}(\exp\{\alpha CK\sum_{i=1}^{n}\nu_i\})}{\exp\{\alpha((n+1)\Delta - T(x))\}}\\
&\quad\leq \frac{\left(2\sqrt{2}\exp\{k^2\alpha^2C^2K^2\gamma^2\Delta\}\right)^n}{\exp\{\alpha((n+1)\Delta - T(x))\}}\\
&\quad= \frac{\exp\{\alpha T(x)\}}{\exp\{\alpha\Delta\}}\exp\left\{\left(k^2\alpha^2C^2K^2\gamma^2 - \alpha + \frac{\log 8}{2\Delta}\right)n\Delta\right\},
\end{aligned}$$

where $\alpha > 0$ is arbitrary and the next to last inequality follows from Lemma 4.3. Choose $\Delta > 0$ (sufficiently large) and $\alpha > 0$ (sufficiently small) such that

$$k^2\alpha^2C^2K^2\gamma^2 - \alpha + \frac{\log 8}{2\Delta} \doteq -\eta < 0.$$

Then

$$\begin{aligned}
P(X^x(s) \notin B^\Delta; 0 \leq s \leq n\Delta) &= P(A_n)\\
&\leq \frac{e^{\alpha T(x)}}{e^{(\alpha-\eta)\Delta}}e^{-\eta(n+1)\Delta},
\end{aligned}$$

for all $n \in \mathbb{N}$. Now let $t \in (0,\infty)$ be arbitrary and n_0 be such that $t \in [n_0\Delta, (n_0+1)\Delta]$. Then

$$\begin{aligned}
P(\tau_{B^\Delta}(x) > t) &= P(X^x(s) \notin B^\Delta; 0 \leq s \leq t)\\
&\leq P(X^x(s) \notin B^\Delta; 0 \leq s \leq n_0\Delta)\\
&\leq \frac{e^{\alpha T(x)}}{e^{(\alpha-\eta)\Delta}}e^{-\eta(n_0+1)\Delta}\\
&\leq \frac{e^{\alpha T(x)}}{e^{(\alpha-\eta)\Delta}}e^{-\eta t}.
\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}(\tau_{B^\Delta}(x)) &= \int_0^\infty P(\tau_{B^\Delta}(x) > t)dt \\ &\le \frac{e^{\alpha T(x)}}{e^{(\alpha-\eta)\Delta}} \int_0^\infty e^{-\eta t}dt \\ &= \frac{e^{\alpha T(x)}}{\eta e^{(\alpha-\eta)\Delta}}.\end{aligned}$$

Recalling that $T(\cdot)$ is a continuous function we have from the above inequality that for all $M \in (0,\infty)$

$$\sup_{x:|x|\le M} \mathbb{E}(\tau_{B^\Delta}(x)) < \infty. \qquad \square$$

LEMMA 4.4. *For $x \in G$ let $\{X^x(t)\}_{t\ge 0}$ be as in Theorem 2.2. Then for all $M \in (0,\infty)$ the family $\{X^x(t); t \ge 0, |x| \le M\}$ is tight.*

PROOF. Let $\Delta > 0$ be large enough so that $B^\Delta \supset A$. Fix $\omega \in \Omega_0(x)$, where $\Omega_0(x)$ is as defined at the beginning of this section. In the rest of the proof we will suppress the dependence of all random variables on ω in the notation. Let

$$S(\Delta) \doteq \{j \in \{1,2,\dots,n-1\} : T(X^x(t)) \le \Delta \qquad \text{for some } t \in [(j-1)\Delta, j\Delta)\}.$$

Define

$$m = \begin{cases} \max\{j : j \in S(\Delta)\}, & \text{if } S(\Delta) \text{ is non empty,} \\ 0, & \text{otherwise.}\end{cases}$$

From Lemma 4.1 we have that

$$T(X^x(n\Delta)) \le T(X^x(m\Delta)) + \sum_{j=m+1}^{n} (KC\nu_j - \Delta). \tag{4.8}$$

Let

$$t \doteq \sup\{s \in [(m-1)\Delta, m\Delta) : T(X^x(s)) \le \Delta\}.$$

If $m > 0$, we have from Lemma 4.1 that

$$\begin{aligned}T(X^x(m\Delta)) &\le (T(X^x(t)) - (m\Delta - t))^+ + KC \sup_{t<s<m\Delta}\left|\int_t^s \sigma(X^x(u))dW(u)\right| \\ &\le \Delta + KC \sup_{t<s<m\Delta}\left|\int_t^s \sigma(X^x(u))dW(u)\right| \\ &\le \Delta + 2KC \sup_{(m-1)\Delta<s<m\Delta}\left|\int_{(m-1)\Delta}^s \sigma(X^x(u))dW(u)\right| \\ &= \Delta + 2KC\nu_m,\end{aligned}$$

where in obtaining the second inequality we have used the fact that for $(m-1)\Delta \le t \le s \le m\Delta$

$$\left|\int_t^s \sigma(X^x(u))dW(u)\right| \le \left|\int_{(m-1)\Delta}^s \sigma(X^x(u))dW(u)\right| + \left|\int_{(m-1)\Delta}^t \sigma(X^x(u))dW(u)\right|.$$

Using this observation in (4.8) we have that

$$\begin{aligned} T(X^x(n\Delta)) &\le T(x) + 2\Delta + \sum_{j=m}^{n} (2KC\nu_j(x) - \Delta) \\ &\le T(x) + 2\Delta + \max_{1\le l\le n} \sum_{j=l}^{n} (2KC\nu_j(x) - \Delta), \end{aligned}$$

where we have written $\nu_j \equiv \nu_j(x)$ in order to explicitly bring out its dependence on x. Hence for $M_0 \in (0, \infty)$,

$$\begin{aligned} P(T(X^x(n\Delta)) \ge M_0) &\le P\left(\max_{1\le l\le n} \sum_{j=l}^{n} (2KC\nu_j(x) - \Delta) \ge M_0 - T(x) - 2\Delta\right) \\ &\le \sum_{l=1}^{n} P\left(2KC\sum_{j=l}^{n} \nu_j(x) \ge M_0 + (n-l-1)\Delta - T(x)\right) \\ &\le \frac{\exp\{\alpha(T(x)+\Delta)\}}{\exp\{\alpha M_0\}} \sum_{l=1}^{n} \frac{\mathbb{E}(\exp\{\alpha 2KC \sum_{j=l}^{n} \nu_j(x)\})}{\exp\{\alpha(n-l)\Delta\}}, \end{aligned}$$

where $\alpha > 0$ is arbitrary. From Lemma 4.3 we now have that

$$\begin{aligned} &P(T(X^x(n\Delta)) \ge M_0) \\ &\quad \le \frac{\exp\{\alpha(T(x)+\Delta)\}}{\exp\{\alpha M_0\}} \sum_{l=1}^{n} \frac{\left(2\sqrt{2}\exp\{8k\alpha^2C^2K^2\gamma^2\Delta\}\right)^{n-l+1}}{\exp\{\alpha(n-l)\Delta\}}, \\ &\quad \le \frac{\exp\{\alpha(2\Delta+T(x))\}}{\exp\{\alpha M_0\}} \sum_{l=1}^{n} \exp\left\{\left(8k\alpha^2C^2K^2\gamma^2\Delta - \alpha\Delta + \frac{\log 8}{2}\right)(n-l+1)\right\}. \end{aligned}$$

Now choose α and Δ so that

$$8k\alpha^2C^2K^2\gamma^2\Delta - \alpha\Delta + \frac{\log 8}{2} = -\theta < 0.$$

Then

$$\begin{aligned} P(T(X^x(n\Delta)) \ge M_0) &\le \frac{\exp\{\alpha(2\Delta+T(x))\}}{\exp\{\alpha M_0\}} \sum_{l=1}^{n} e^{-\theta(n-l)} \\ &\le \frac{\exp\{\alpha(2\Delta+T(x))\}}{\exp\{\alpha M_0\}(1-\exp\{-\theta\})}. \end{aligned}$$

Hence for all $M, M_0 \in (0, \infty)$

$$\sup_{n\in\mathbb{N},|x|\le M} P(T(X^x(n\Delta)) \ge M_0) \le \frac{\exp\{\alpha(2\Delta + \frac{4K^2}{\delta}M)\}}{\exp\{\alpha M_0\}(1 \exp\{-\theta\})}.$$

From Lemma 3.1(ii) it now follows that

$$\sup_{n\in\mathbb{N},|x|\le M} P(X^x(n\Delta) \ge M_0) \le \frac{\exp\{\alpha(2\Delta + \frac{4K^2}{\delta}M)\}}{\exp\{\alpha c M_0\}(1 - e^{-\theta})}. \tag{4.9}$$

Now let $t \in [n\Delta, (n+1)\Delta]$ and consider the process $\{\tilde{Y}(t)\}_{0\le t\le\Delta}$ defined in (4.2) with u there replaced by $n\Delta$. For each n it follows from (4.3) that

$$|X^x(t) - \tilde{Y}(t - n\Delta)| \le K\nu_n(x).$$

Define a function $\tilde{b} : G \to \mathbb{R}^k$ which agrees with b off A and satisfies Condition 2.3 with $A = \emptyset$ as well as equations (2.2). Also define

$$Y^*(t) \doteq \Gamma\left(X^x(n\Delta) + \int_0^\cdot \tilde{b}(X^x(s + n\Delta))ds\right)(t).$$

Clearly

$$L \doteq \sup_{x\in G} |b(x) - \tilde{b}(x)| < \infty.$$

Furthermore

$$\begin{aligned}|\tilde{Y}(t - n\Delta)| &\le |\tilde{Y}(t - n\Delta) - Y^*(t - n\Delta)| + |Y^*(t - n\Delta)| \\ &\le KL\Delta + K|X^x(n\Delta)|,\end{aligned}$$

where in obtaining the last inequality we have used the Lipschitz property of Γ and Theorem 3.2. Combining the above observations we have that

$$|X^x(t)| \le K(\nu_n(x) + |X^x(n\Delta)|) + KL\Delta.$$

Therefore, for $M_0 \in (0, \infty)$, any n, and $t \in [n\Delta, n\Delta + \Delta]$,

$$\begin{aligned}P(|X^x(t)| \ge M_0) \le P&\left(\nu_n(x) \ge \tfrac{M_0 - KL\Delta}{2K}\right) \\ &+ P\left(|X^x(n\Delta)| \ge \tfrac{M_0 - KL\Delta}{2K}\right).\end{aligned} \tag{4.10}$$

Clearly, the family $\{|\nu_n(x)|; n \ge 1, |x| \le M\}$ is tight. Now let $\eta > 0$ be arbitrary. Choose $M_0 \in (0, \infty)$ such that

$$\sup_{n\in\mathbb{N},|x|\le M} P\left(|\nu_n(x)| \ge \frac{M_0 - KL\Delta}{2K}\right) \le \frac{\eta}{2}$$

and

$$\frac{\exp\{\alpha(2\Delta + \frac{4K^2}{\delta}M)\}}{\exp\{\frac{\alpha c(M_0 - KL\Delta)}{2K}\}(1 - e^{-\theta})} \le \frac{\eta}{2}.$$

Then from (4.9) and (4.10) we have that

$$\sup_{t\geq 0, |x|\leq M} P(|X^x(t)| \geq M_0) \leq \eta.$$

Since $\eta > 0$ is arbitrary, we have the result. $\square$

From Theorem 4.1, Lemma 4.4 and Condition 2.4 the proof of positive recurrence and the existence and uniqueness of an invariant measure for $\{X^x(t)\}_{t\geq 0}$ is standard (cf. [9]). However for the sake of completeness we present the proof below.

PROOF OF THEOREM 2.2. Denote the measure induced by $\{X^x(\cdot)\}$ on $C([0,\infty) : G)$ by P_x, where $C([0,\infty) : G)$ is the space of G valued continuous functions defined on the nonnegative real line. In arguments that are presented below, it will be convenient to let initial conditions be defined through conditioning, rather than though the superscript as in X^x. As a consequence, instead of X^x we will work with the canonical process $\xi(\cdot)$ on $C([0,\infty) : G)$, and the canonical filtration which we denote by $\{\mathscr{F}_t\}$. Finally, the expectation operator corresponding to the probability measure P_x will be denoted by $\mathbb{E}_x$. Given a compact set $B \subset G$, let

$$\tilde{\tau}_B \doteq \inf\{t : \xi(t) \in B\}.$$

In order to show positive recurrence, we need to show that if S is an arbitrary compact set in G with positive Lebesgue measure then for all $x \in G$, $\mathbb{E}_x \tilde{\tau}_S < \infty$. Let B^Δ be as in Theorem 4.1, and let $r \in (0,\infty)$ be such that $B^\Delta \subset \{x : |x| \leq r\}$. Then from Theorem 4.1 we have that for all $C \in (0,\infty)$

$$\sup_{x:|x|\leq C} \mathbb{E}_x(\tilde{\tau}_{B_r}) < \infty, \tag{4.11}$$

where $B_r \doteq \{x \in G : |x| \leq r\}$. From the uniform non degeneracy assumption (Condition 2.4), we have (cf. [12])

$$p(S) \doteq \inf_{x\in B_r} P_x(\xi(1) \in S) > 0.$$

Furthermore, the Feller property of $\{X^x(\cdot)\}$ implies that the family $\{X^x(t) : x \in B_r, 0 \leq t \leq 1\}$ is tight, and so there exists $M \in (0,\infty)$ such that

$$\inf_{x\in B_r} P_x(\xi(1) \in S \text{ and } |\xi(t)| \leq M \qquad \text{for all } t \in [0,1]) \geq \frac{p(S)}{2}. \tag{4.12}$$

Let $C \in (M,\infty)$ be fixed, and define

$$\hat{\tau} \doteq \inf\{t : |\xi(t)| \geq C\}$$

and $\tilde{\tau} \doteq \min\{1, \hat{\tau}, \tilde{\tau}_S\}$. If $y \in B_r$, then by the strong Markov property

$$\begin{aligned} \mathbb{E}_y(\tilde{\tau}_S) &= \mathbb{E}_y\left(\mathbb{E}_y\left(\tilde{\tau}_S \mid \mathscr{F}_{\tilde{\tau}}\right)\right) \\ &\leq \mathbb{E}_y\left(\tilde{\tau} + \mathbb{E}_{\xi(\tilde{\tau})}(\tilde{\tau}_S)\right) \\ &\leq 1 + \mathbb{E}_y\left(\mathbb{E}_{\xi(\tilde{\tau})}(\tilde{\tau}_S)\right). \end{aligned} \tag{4.13}$$

Now define

$$\Lambda \doteq \left\{ \xi(\cdot) \in C([0,\infty) : G) : \sup_{0 \le t \le 1} |\xi(t)| \le M \text{ and } \xi(1) \in S \right\}.$$

Since $E_{\xi(\tilde{\tau})}(\tilde{\tau}_S) = 0$ w.p.1 if $\xi \in \Lambda$, for $y \in B_r$,

$$\begin{aligned}\mathbb{E}_y\left(\mathbb{E}_{\xi(\tilde{\tau})}(\tilde{\tau}_S)\right) &= \mathbb{E}_y\left(\mathscr{I}_\Lambda(\xi)\mathbb{E}_{\xi(\tilde{\tau})}(\tilde{\tau}_S)\right) + \mathbb{E}_y\left(\mathscr{I}_{\Lambda^c}(\xi)\mathbb{E}_{\xi(\tilde{\tau})}(\tilde{\tau}_S)\right) \\ &= \mathbb{E}_y\left(\mathscr{I}_{\Lambda^c}(\xi)\mathbb{E}_{\xi(\tilde{\tau})}(\tilde{\tau}_S)\right).\end{aligned} \tag{4.14}$$

Next, fix $z \in G$ such that $|z| \le C$. Then

$$\mathbb{E}_z(\tilde{\tau}_S) = \mathbb{E}_z(\tilde{\tau}_{B_r} + (\tau_S - \tilde{\tau}_{B_r})).$$

Once more using the strong Markov property, we have

$$\begin{aligned}\mathbb{E}_z(\tilde{\tau}_S) &\le \mathbb{E}_z\left(\tilde{\tau}_{B_r} + \sup\nolimits_{x \in B_r} \mathbb{E}_x(\tilde{\tau}_S)\right) \\ &\le \sup\nolimits_{z:|z|\le C} \mathbb{E}_z(\tilde{\tau}_{B_r}) + \sup\nolimits_{x \in B_r} \mathbb{E}_x(\tilde{\tau}_S).\end{aligned} \tag{4.15}$$

Observing that $|\xi(\tilde{\tau})| \le C$ and combining (4.14) and (4.15) we have that for $y \in B_r$

$$\mathbb{E}_y\left(\mathbb{E}_{\xi(\tilde{\tau})}(\tilde{\tau}_S)\right) \le \left(\sup_{z:|z|\le C} \mathbb{E}_z(\tilde{\tau}_{B_r}) + \sup_{x \in B_r} \mathbb{E}_x(\tilde{\tau}_S) \right) P_y(\Lambda^c). \tag{4.16}$$

From (4.12), (4.14) and (4.16) it now follows that

$$\sup_{y \in B_r} \mathbb{E}_y(\tilde{\tau}_S) \le 1 + \sup_{z:|z|\le C} \mathbb{E}_z(\tilde{\tau}_{B_r}) + \left(1 - \frac{p(S)}{2}\right) \sup_{x \in B_r} \mathbb{E}_x(\tilde{\tau}_S).$$

Thus,

$$\sup_{y \in B_r} \mathbb{E}_y(\tilde{\tau}_S) \le \frac{2}{p(S)} \left\{ 1 + \sup_{z:|z|\le C} \mathbb{E}_z(\tilde{\tau}_{B_r}) \right\}.$$

A final application of the strong Markov property now yields that for $x \in G$,

$$\begin{aligned}\mathbb{E}_x(\tilde{\tau}_S) &\le \mathbb{E}_x(\tilde{\tau}_{B_r}) + \sup_{y \in B_r} \mathbb{E}_y(\tilde{\tau}_S) \\ &\le \mathbb{E}_x(\tilde{\tau}_{B_r}) + \frac{2}{p(S)} \left\{ 1 + \sup_{z:|z|\le C} \mathbb{E}_z(\tilde{\tau}_{B_r}) \right\} \\ &< \infty,\end{aligned}$$

where the last inequality follows from Theorem 4.1 and (4.11). This completes the proof of positive recurrence.

Finally we consider the existence and uniqueness of invariant measures. From Lemma 4.4 we have that the family of measures $\{\mu_t; t \ge 1\}$ defined by

$$\mu_t(B) \doteq \frac{1}{t} \int_0^t P(X^x(s) \in B)\, ds$$

is tight. Since the Markov process $\{X^x(t)\}$ is Feller we have that any weak limit of $\{\mu_t\}$ is an invariant measure. (See, e.g., the proof of Theorem 4.1.21, Chapter I, [20]). Finally uniqueness follows as in [12, 9] in view of Condition 2.4. □

REMARK 4.1. The Lipschitz and growth condition (Condition 2.2) on b and σ are essentially assumed to guarantee a unique solution to the constrained diffusion process (1.1) which is Feller-Markov. The conclusion of Theorem 2.2 continues to hold with the same proof if Condition 2.2 is replaced by the assumption that (2.3) holds for some $\gamma \in (0, \infty)$, b is locally bounded and (1.1) has a unique weak solution with continuous paths for every $x \in G$ and the solution is Feller-Markov.

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