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Deterministic and stochastic differential inclusions with multiple surfaces of discontinuity

Rami Atar · Amarjit Budhiraja · Kavita Ramanan

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Abstract We consider a class of deterministic and stochastic dynamical systems with discontinuous drift f and solutions that are constrained to live in a given closed domain G in $\mathbb{R}^n$ according to a constraint vector field $D(\cdot)$ specified on the boundary ∂G of the domain. Specifically, we consider equations of the form

$$\phi = \theta + \eta + u, \quad \dot{\theta}(t) \in F(\phi(t)), \quad \text{a.e. } t$$

for u in an appropriate class of functions, where η is the "constraining term" in the Skorokhod problem specified by (G, D) and F is the set-valued upper semicontinuous envelope of f. The case $G = \mathbb{R}^n$ (when there is no constraining mechanism) and u is absolutely continuous corresponds to the well known setting of differential

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R. Atar
Department of Electrical Engineering, Technion, Haifa 32000, Israel
e-mail: atar@ee.technion.ac.il

A. Budhiraja (✉)
Department of Statistics and Operations Research, University of North Carolina, Chapel Hill, NC 27599, USA
e-mail: budhiraj@email.unc.edu

K. Ramanan
Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA
e-mail: kramanan@math.cmu.edu

inclusions (DI). We provide a general sufficient condition for uniqueness of solutions and Lipschitz continuity of the solution map, in the form of existence of a Lyapunov set. Here we assume (i) G is convex and admits the representation $G = \cup_i \overline{C_i}$, where $\{C_i, i \in \mathbb{I}\}$ is a finite collection of disjoint, open, convex, polyhedral cones in $\mathbb{R}^n$, each having its vertex at the origin; (ii) $f = b + f^c$ is a vector field defined on G such that b assumes a constant value on each of the given cones and f^c is Lipschitz continuous on G; and (iii) D is an upper semicontinuous, cone-valued vector field that is constant on each face of ∂G. We also provide existence results under much weaker conditions (where no Lyapunov set condition is imposed). For stochastic differential equations (SDE) (possibly, reflected) that have drift coefficient f and a Lipschitz continuous (possibly degenerate) diffusion coefficient, we establish strong existence and pathwise uniqueness under appropriate conditions. Our approach yields new existence and uniqueness results for both DI and SDE even in the case $G = \mathbb{R}^n$. The work has applications in the study of scaling limits of stochastic networks.

Keywords Discontinous drift · Ordinary differential equations · Differential inclusions · Stochastic differential equations · Stochastic differential inclusions · Reflected diffusions · Skorokhod map · Skorokhod problem

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1 Introduction, problem description and main results

1.1 General framework

This paper introduces and analyses a framework for dynamical systems with multiple intersecting surfaces of discontinuity, that includes differential inclusions (DI) and stochastic differential equations (SDE) with discontinuous drift, and yields new existence and uniqueness results in both fields. For both types of equations, the solutions are required to be constrained at all times to a given set $G \subset \mathbb{R}^n$, that is the closure of an open connected set. In the case where G is a strict subset of $\mathbb{R}^n$, our formulation of SDE corresponds to stochastic differential equations with reflection (SDER) that have a discontinuous drift. Our treatment also covers the case $G = \mathbb{R}^n$ in which there is no constraint (or reflection). Even in the absence of constraint, uniqueness of solutions for formulations in which the velocity vector field is discontinuous has been studied mostly when the vector field has a single, smooth surface of discontinuity [1,8,18], with the exception of [15]. However, the conditions for uniqueness in [15] are not given explicitly in terms of the problem data, and are not natural for the study of SDE (see Sect. 1.2.2 for further discussion of this issue). We aim in this paper at a setting that goes beyond a single, smooth surface of discontinuity and obtain explicit sufficient conditions on the problem data for existence and uniqueness of solutions that are applicable in both the deterministic and stochastic settings.

The general form of the dynamical systems of interest, in the absence of constraints (i.e. when $G = \mathbb{R}^n$), is

$$\phi = \theta + u, \qquad \dot{\theta}(t) \in F(\phi(t)) \quad a.e.\ t \tag{1.1}$$

for u in an appropriate class of functions and a suitable set-valued velocity field F. We refer to this formulation as the *discontinuous media problem* (DMP). For the case where G is a proper subset of $\mathbb{R}^n$, a natural framework for constraining the dynamics to G is through the notion of the Skorokhod problem (SP) [6,9,19, 21,22]. Roughly speaking, given a domain and a *constraint vector field* D on its boundary, the SP associates to any given trajectory ψ a version that is restricted to the domain. Such a version is obtained from ψ by adding a term that is locally of bounded variation, and acts as a "singular drift" on the boundary, along directions determined by D. The relationship between the DMP and the SP is, in fact, more intimate than might first be apparent. Indeed, it turns out that a strategy used for establishing uniqueness of solutions for the SP can be adapted to the DMP. Moreover, a formulation that embraces both discontinuity aspects turns out to be natural. In this formulation, which we refer to as the *constrained discontinuous media problem* (CDMP), G is a proper subset of $\mathbb{R}^n$ with a constraint vector field D defined on its boundary, and F is a velocity vector field defined on G, in a fashion analogous to the DMP. By adopting the perspective that the constraining action of the SP essentially gives rise to a discontinuity in the drift across the boundary of the domain, one can view CDMP dynamical systems as featuring two kinds of discontinuities—'interior' and 'boundary' discontinuities, corresponding to the drift and the constraint directions, respectively. Some two-dimensional examples of domains, velocity and direction fields for the DMP, SP and CDMP are depicted in Fig. 1. Rigourous definitions of the DMP and CDMP are given in Sect. 1.3.

We establish existence of solutions to the CDMP under rather general conditions on (G, D, F) (see Sect. 1.5 for a summary of the main results). For uniqueness, we restrict ourselves to the class of so-called "polyhedral CDMPs" for which G is a convex, polyhedral cone in $\mathbb{R}^n$ with vertex at the origin, with a piecewise constant constraint vector field D defined on its boundary, and a velocity vector field in the interior that can be represented as the sum of a Lipschitz continuous function (throughout, by the term 'Lipschitz' we mean 'globally Lipschitz') and a piecewise constant function that

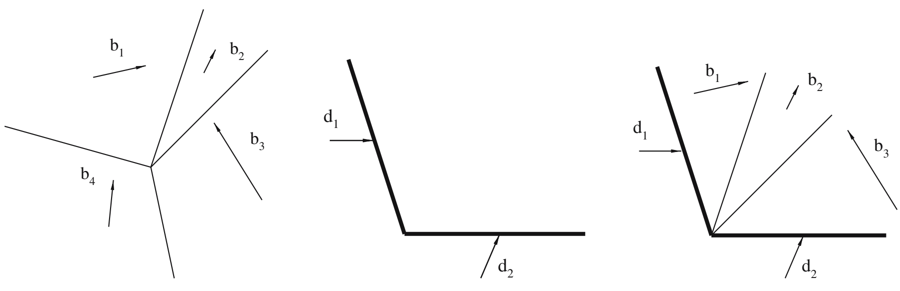


Fig. 1 **a** A discontinuous media problem with velocity vector field described by cones C_i and vectors b_i. **b** A Skorokhod problem with constraint directions d_i on the faces of a polyhedral domain. **c** A constrained discontinuous media problem that combines a discontinuous velocity field and constraint

is discontinuous across multiple hyperplanes that pass through the origin. We present a sufficient condition (Assumption 4) for uniqueness, stated in terms of the existence of a suitable Lyapunov set. Similar Lyapunov set conditions have been very useful in the study of uniqueness questions for SPs [9,11]. We find that a joint Lyapunov set that handles both "interior" and "boundary" discontinuities is a natural way to treat uniqueness for the CDMP. To illustrate this point, we show in Example 2.12 that even when there is uniqueness for a given SP and for a given DMP, the CDMP that results by combining them may admit multiple solutions. Our approach, which yields uniqueness results for the CDMP, is also useful for establishing pathwise uniqueness for SDE with discontinuous drift whose solutions are constrained to a domain G (sometimes referred to as SDER).

A framework that treats both deterministic and stochastic dynamical systems and allows for both interior and boundary discontinuities has been developed in Cepa [5]. The conditions in [5] however, are quite different and do not cover our results. We comment on the setting of [5] and its relation to the current paper in Sect. 1.2.2.

In both the DMP and CDMP formulations, one could consider uniqueness for problems in which the hyperplanes of discontinuity of the vector fields (and the hyperplanes defining the boundary of the domain) do not all intersect at a single point. However, it is expected that relatively standard localisation arguments could be invoked to extend the results of this paper to deal with such settings. Indeed, analogous extensions have been successfully carried out in the pure SP setting [7,19]. One could, of course, also consider smooth surfaces, in place of hyperplanes, and more general piecewise continuous vector fields, but our setting is a natural first step towards these broader formulations. Finally, we emphasise that there are many problems arising in applications that have the "polyhedral" structure described above (see Example 2.14). Indeed, functional strong law of large numbers and functional central limit approximations of pure jump processes arising in stochastic networks give rise to CDMPs and, respectively, SDERs of this form. Applications of the results of this paper to these problems will be considered in future work.

In the case when u is absolutely continuous, our formulation can be viewed as a constrained differential inclusion. Thus, to set our work in perspective, in Sect. 1.2 below we review some relevant results on constrained differential inclusions. For the setting and statements of the main results of the paper, the reader may safely skip to Sect. 1.3. The generalisation to u that are not absolutely continuous, as embodied in the CDMP, is essential for the study of corresponding SDE and SDER with discontinuous drift. In Sect. 1.3, we provide a rigorous definition of the CDMP—the special class of polyhedral CDMPs is described in Sect. 1.4. An outline of the paper and its main results is provided in Sect. 1.5.

1.2 Some background on constrained differential inclusions

Our analysis of SDE (SDER) is pathwise and based on the DMP (respectively, CDMP). Since DMPs are closely related to DI, we begin by discussing DI in general, and then describe our particular setting. Our aim here is not to provide a complete survey of results on DI, but merely to place our work in context.

1.2.1 Description of constrained differential inclusions

The classical theory of ordinary differential equations is concerned with dynamical systems of the type

$$\dot{\phi}(t) = \tilde{f}(t, \phi(t)), \quad t \geq 0 \tag{1.2}$$

where $\tilde{f}(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function of t and x. It is well understood that when $\tilde{f}$ is discontinuous, an appropriate framework for the analysis of (1.2) is through the theory of differential inclusions [3,15]. In this theory one considers set-valued functions $\tilde{F}$ and relations of the form

$$\dot{\phi}(t) \in \tilde{F}(t, \phi(t)), \quad a.e.\ t. \tag{1.3}$$

In particular, when $\tilde{f}$ is piecewise continuous, $\tilde{F}(t, x)$ is constructed from $\tilde{f}(t, x)$ by a convexification procedure (cf. [3]). A special case of (1.3) is

$$\dot{\phi}(t) \in F(\phi(t)) + v(t), \tag{1.4}$$

where for each x, $F(x)$ is a nonempty, bounded, closed, convex set and the set-valued function F is upper semicontinuous (u.s.c.) and $v : \mathbb{R}_+ \to \mathbb{R}^n$ is a given function. We are in fact interested in the more general integral equation (1.1) with u being an arbitrary RCLL (right continuous with finite left limits) trajectory. Note that (1.1) reduces to (1.4) when u is absolutely continuous and $v = \dot{u}$. This integral formulation is natural for the study of SDEs of the form

$$dX_t = \alpha_t\, dt + \sigma(X_t)\, dW_t, \quad \alpha_t \in F(X_t), \ \text{a.e. } t,$$

where W is a standard Brownian motion and σ is a diffusion coefficient. Such SDE model autonomous DI of the form $\dot{\phi}(t) \in F(\phi(t))$, perturbed by *noise*, and are sometimes referred to as *multivalued* SDE (see, for example [5]) or as *stochastic differential inclusions*. The general form of SDE studied in this paper can be found in Sect. 3.2.

As we already mentioned, it is also of interest to consider a formulation in which F is defined on the closure G of a domain that is a proper subset of $\mathbb{R}^n$, and the SP is invoked to define solutions that are constrained to G. Here we only describe the constrained analogue of (1.4); the more general setting corresponding to the constrained form of (1.1) is presented in Sect. 1.3. For a given set A, we will denote its interior by A^o. Let D be a set-valued, u.s.c. function defined on G such that for each $x \in G$, $D(x)$ is a convex cone and for $x \in G^o$, $D(x) = \{0\}$. Define for $x \in G$,

$$F(x) + D(x) = \left\{\beta \in \mathbb{R}^n : \beta = \beta_1 + \beta_2, \beta_1 \in F(x), \ \beta_2 \in D(x)\right\}.$$

Constrained DI studied in this work (as a special case of more general dynamical systems) take the form

$$\dot{\phi}(t) \in F(\phi(t)) + D(\phi(t)) + v(t), \quad \phi(t) \in G. \tag{1.5}$$

1.2.2 Summary of some prior results

The results of this paper provide, in particular, conditions for existence and uniqueness of solutions to constrained DI described in the last section. To the best of our knowledge [5] is the only paper prior to this work that treats general existence and uniqueness results for (1.5) in the setting where F is a nontrivial set-valued function and $G \subsetneq \mathbb{R}^n$. Specifically, the paper [5] establishes existence and uniqueness of solutions under the assumption that the velocity vector field F is the negative of a maximal monotone operator (see also Theorem 3.2.1 of [3]). This includes, in particular, the case when $-F$ is the subdifferential of a proper, lower semicontinuous, convex function V, and also allows for the case when G is a strict convex subset of $\mathbb{R}^n$ with normal directions of constraint on the boundary.

Other prior results in this domain fall mainly into two categories: the case when there is either no boundary discontinuity (so that $G = \mathbb{R}^n$), leading to a DI, or there is no interior discontinuity (so that F is single-valued and continuous), which we refer to as constrained ODE. We now provide a brief description of these results—as elaborated below, even in the case when there is no boundary discontinuity, but the interior velocity field has multiple intersecting surfaces of discontinuity, currently existing conditions for uniqueness are not completely satisfactory.

A. DI on $\mathbb{R}^n$ (*where* $D = \{0\}$).

1. *Existence.* In this case, existence of solutions (on a suitable time interval) to the DI (1.4) follows from standard theorems (see, for example, Theorem 7.1 in [15] for the case where each F is locally bounded and u.s.c., and v is u.s.c.; the generalisation to the case when v is locally integrable is straightforward).
2. *Uniqueness.* Other than the results of [3,5] discussed above, this seems to have been studied in depth only in the following cases.

   (i) When F is obtained from the convexification of a function f whose discontinuities only occur across isolated surfaces, necessary and sufficient conditions for uniqueness can be found in [15, Sect. 2.10]; also see [1,18] for a related analysis. In particular, consider the following case concerning a single hyperplane of discontinuity: let $f(y)$ take the value b_1 for $\langle y, e_1 \rangle < 0$ and b_2 for $\langle y, e_1 \rangle > 0$, where $b_1, b_2 \in \mathbb{R}^n$ are constant vectors. Then the condition

   $$\langle b_1 - b_2, e_1 \rangle > 0 \tag{1.6}$$

   is sufficient for uniqueness of solutions to the DI (1.4) with initial condition $\phi(0) = x$ (see, e.g. [15, Sect. 2.10, Theorem 2]). We comment that this setting is not covered by the framework in [3,5] since the maximal monotonicity of $-F$ requires that $b_1 - b_2$ be a non-negative multiple of e_1.

   (ii) In the setting where several surfaces of discontinuities of f have a common point of intersection, Filippov established uniqueness of solutions to a large class of DI of the form (1.3) under the requirement that the solution not pass from one surface of discontinuity to another an infinite number of times in a finite time interval (see [15, Theorem 10.4] for a precise statement). This condition is not specified fully in terms of the problem data and is thus hard to verify. Moreover, it is far from being necessary [15, p. 116] and is not a natural condition for extensions to the setting of SDE. In particular, the more general case of a locally integrable function v is not covered by Filippov's conditions.

B. Constrained ODE on $G \subsetneq \mathbb{R}^n$

Consider now the case where $F(x) = \{f(x)\}$ for every $x \in G$. When f is the constant function, this coincides with the SP, which has been studied extensively in past works [2,4,6,9,11,16,19,22]. Thus, below we only summarise results when f has non-trivial state-dependence.

1. *Existence.* It was shown in [9] that solutions to (1.5) exist in the case where f is Lipschitz and there exists a Lipschitz continuous projection operator (see Assumption 1) associated with the SP (G, D). Other papers that treat existence include [2,19,22].
2. *Uniqueness.* For general Lipschitz continuous f, uniqueness has been treated in [2,9,19,22], with general oblique directions of constraint on the boundary. In particular, a Picard iteration argument was used to establish pathwise uniqueness of a class of SDERs in [2], and applied to establish uniqueness of constrained ODEs in [9].

1.3 The constrained discontinuous media problem

Let G be the closure of an open, connected set in $\mathbb{R}^n$ and D be a set-valued vector field defined on the boundary ∂G such that for every $x \in \partial G$, $D(x)$ is a closed, convex cone. Extend the definition of D to G by setting $D(x) = \{0\}$ for all $x \in G^o$. Let F be a set-valued vector field that maps points in G to subsets of $\mathbb{R}^n$. We will refer to D and F as the constraint vector field and, respectively, the interior velocity or drift vector field. Roughly speaking, we seek solutions to dynamical systems that are governed by the velocity vector field F in the interior of G and constrained to lie in G according to the constraint vector field D and driven by a given path u. We refer to them as solutions to the constrained discontinuous media problem (CDMP) associated with G, D, F and u. A precise formulation is as follows. Let $\mathcal{D}[0, T]$ be the space of $\mathbb{R}^n$-valued functions on $[0, T]$, that are RCLL, with the usual Skorokhod J_1 topology. The total variation of $\phi \in \mathcal{D}[0, T]$ over $[0, t]$ will be denoted by $|\phi|(t)$. We denote the space of $\mathbb{R}^n$-valued, integrable functions on $[0, T]$ by $\mathcal{L}^1[0, T]$, the space of $\mathbb{R}^n$-valued, bounded and measurable functions on $[0, T]$ by $\mathcal{BM}[0, T]$ and the space of $\mathbb{R}^n$-valued absolutely continuous functions on $[0, T]$ by $\mathcal{AC}[0, T]$.

Definition 1.1 (Constrained discontinuous media problem) Let $u \in \mathcal{D}[0, T]$. We say that $\phi \in \mathcal{D}[0, T]$ is a solution of the CDMP associated with G, D, F and u, and write $\phi \in \mathcal{M}(G, D, F, u)$, if the following properties hold:

(i) $\phi(t) \in G$ for all $t \in [0, T]$;

(ii) there exists $\eta \in \mathcal{D}[0, T]$ and $\theta \in \mathcal{AC}[0, T]$ such that

$$\phi(t) = u(t) + \theta(t) + \eta(t), \quad t \in [0, T];$$

(iii) there exists $\alpha \in \mathcal{L}^1[0,T]$ such that $\alpha(t) \in F(\phi(t))$, a.e. $t \in [0,T]$ and for $t \in [0,T]$,

$$\theta(t) = \int_0^t \alpha(s)\,ds;$$

(iv) $|\eta|(T) < \infty$ and

$$|\eta|(t) = \int_{[0,t]} 1\!\!1_{\{\phi(s)\in\partial G\}}\,d|\eta|(s), \qquad t \in [0,T];$$

(v) there exists $\gamma \in \mathcal{BM}[0,T]$ such that $\gamma(s) \in D(\phi(s))$ for $d|\eta|$-a.e. $s \in [0,T]$ and for $t \in [0,T]$,

$$\eta(t) = \int_{[0,t]} \gamma(s)\,d|\eta|(s).$$

In the special case where $F(x) = \{0\}$ for all $x \in G$, we refer to the CDMP associated with G, D, F and u as the SP associated with G, D and u. Similarly, in the case where $G = \mathbb{R}^n$ and $D = \{0\}$, we refer to the CDMP associated with G, D, F and u as the DMP associated with F and u.

1.4 Polyhedral CDMPs

For uniqueness we focus on a special class of CDMPs. To this end we introduce the following notation. A subset of $\mathbb{R}^n$ is said to be a closed (open) *half space* if it takes the form $\{x \in \mathbb{R}^n : \langle x, \nu\rangle \geq 0\}$ (respectively, $\{x \in \mathbb{R}^n : \langle x, \nu\rangle > 0\}$), for some $\nu \neq 0$. A set that is either equal to $\mathbb{R}^n$ or is a subset of $\mathbb{R}^n$ given by a finite intersection of half spaces is said to be a (convex) *polyhedral cone*. Given a set $A \subset \mathbb{R}^n$ we denote its closure by $\overline{A}$ and its closed, convex hull by $\text{co}(A)$.

Let f be a piecewise continuous vector field defined on a closed, convex, polyhedral cone $G \subseteq \mathbb{R}^n$. Moreover, suppose that G is the closure of the set $\mathcal{O}$ of continuity points of f, and that $\mathcal{O}$ is the union of a finite number of disjoint, open, convex, polyhedral cones C_i, $i = 1, \ldots, I$, with $f = b + f^c$ on $\mathcal{O}$, where f^c is Lipschitz continuous on G and b is equal to the vector $b_i \in \mathbb{R}^n$ on the cone C_i, for each i. Define $\mathbb{I} \doteq \{1, \ldots, I\}$, let

$$\mathcal{C} \doteq \{C_i, i \in \mathbb{I}\} \tag{1.7}$$

denote the partition of $\mathcal{O}$ into convex polyhedral cones and let the corresponding collection of drift vectors be given by

$$\mathcal{B} \doteq \{b_i, i \in \mathbb{I}\}. \tag{1.8}$$

Note that by definition $b : \mathcal{O} \to \mathbb{R}^n$ satisfies

$$b(x) = b_i \qquad \text{for } x \in C_i,\ i \in \mathbb{I}. \tag{1.9}$$

The interior velocity field F will be obtained as a convexification of f, given as the following set-valued function from G to convex, compact subsets of $\mathbb{R}^n$: for $x \in G$,

$$F(x) \doteq \left\{ \sum_{i \in I(x)} \alpha_i b_i + f^c(x) : \sum_{i \in I(x)} \alpha_i = 1, \alpha_i \in [0,1], i \in \mathbb{I} \right\}, \tag{1.10}$$

where $I(x) \doteq \{i \in \mathbb{I} : x \in \overline{C}_i\}$. It is easy to verify that the graph of F is closed; indeed, if $x_n \to x \in G$ and $u_n \in F(x_n)$ converges to u as $n \to \infty$, then $u \in F(x)$ since $F(\cdot)$ is u.s.c.

By assumption, G is given as the intersection of K closed half spaces, $G_1, \dots, G_K$, where $G_k \doteq \{x \in \mathbb{R}^n : \langle x, n_k \rangle \geq 0\}$, $k = 1, \dots, K$. Let $\mathbb{K} \doteq \{1, \dots, K\}$. Associated with each half space G_k, $k \in \mathbb{K}$, we are given a vector $d_k \in \mathbb{R}^n$ such that

$$\langle d_k, n_k \rangle > 0, \ \text{ for all } k \in \mathbb{K}. \tag{1.11}$$

Define

$$D(x) \doteq \left\{ \sum_{k \in K(x)} \gamma_k d_k : \ \gamma_k \geq 0,\ k \in K(x) \right\}, \tag{1.12}$$

where $K(x) \doteq \{k \in \mathbb{K} : \langle x, n_k \rangle = 0\}$. The CDMP associated with the data (G, D, F), constructed from d_k, n_k, $k = 1, \dots, K$ and $\mathcal{C}$, $\mathcal{B}$ and f^c as above, will be referred to as a polyhedral CDMP.

Remark 1.2 A standard measurable selection argument [14, Corollary 10.3, Appendix] shows that with ϕ and α as in the definition of CDMP, and with (G, D, F) as above, there exist $\lambda_i \in \mathcal{BM}([0, T] : [0, 1])$, $i \in \mathbb{I}$, such that for a.e. $t \in [0, T]$, $\sum_{i \in \mathbb{I}} \lambda_i(t) = 1$, $\lambda_i(t) 1\!\!1_{\{\phi(t) \notin \overline{C}_i\}} = 0$ and $\alpha(t) = \sum_{i \in \mathbb{I}} \lambda_i(t) b_i + f^c(\phi(t))$.

1.5 Main results and outline of paper

In Sect. 2 we provide general sufficient conditions for the existence and uniqueness of solutions to CDMPs. In Sect. 3 we introduce the related class of stochastic differential inclusions, and stochastic differential equations with reflection that have discontinuous drift, and establish strong existence and pathwise uniqueness of solutions for these SDE. A summary of the results obtained in these sections is as follows.

(a) *Existence.* Theorem 2.3 establishes existence of solutions to the CDMP (G, D, F) for RCLL u under the assumption that F is a uniformly bounded u.s.c. set-valued function on G, with each $F(x)$, $x \in G$, being a convex, compact subset of $\mathbb{R}^n$, and that the SP (G, D) satisfies Assumptions 1 and 2. These assumptions on the SP are quite mild (see Remark 2.1).

(b) *Uniqueness.* Uniqueness is studied for the class of polyhedral CDMPs. We treat it in two main steps. In Sect. 2 we consider the special case $f^c = 0$ while the general setting ($f^c \neq 0$) is covered as a particular case (with $\sigma = 0$) of the pathwise uniqueness result for SDE established in Theorem 3.4. For the case $f^c = 0$, Theorem 2.9 provides a general sufficient condition, namely the validity of Assumptions 3 and 4, for uniqueness of solutions and regularity of the solution map. Combining the uniqueness and existence results, Theorem 2.11 shows that if the polyhedral CDMP satisfies Assumptions 1, 2, 3 and 4, then the solution mapping is well-defined and Lipschitz continuous on all of $\mathcal{D}[0, T]$. For the case $G = \mathbb{R}^n$, Theorems 2.9 and 2.11 allow for multiple surfaces of discontinuity as well as a general class of input functions u, and thus address settings not covered by the results described in Sect. 1.2.2 on uniqueness of differential inclusions.

(c) *SDE.* We obtain new existence and uniqueness results for SDE (SDER) with discontinuous drift. We restrict ourselves to the case where (G, D, F) is associated with a polyhedral CDMP, as described in Sect. 1.4. In Sect. 3.2 we introduce SDE for which the velocity vector field F serves as the drift coefficient, and the diffusion coefficient is Lipschitz continuous and possibly degenerate. We further allow a progressively measurable RCLL input on the right hand side of the equation. In Theorem 3.4 we establish strong existence and pathwise uniqueness. Here too, we consider a formulation that allows for a constraint vector field, giving rise to constrained diffusions. See Remark 3.5 for comparison of our results with prior results on strong solutions to SDE with discontinuous coefficients.

2 Existence and uniqueness of solutions

We establish sufficient conditions for existence of solutions to a large class of CDMPs in Sect. 2.1 and for uniqueness of solutions to the class of polyhedral CDMPs in Sect. 2.2. Section 2.3 contains some concrete examples of CDMPs that satisfy these assumptions.

2.1 Existence of solutions

We first impose two natural conditions (Assumptions 1 and 2) on the constraint data (G, D) associated with the CDMP (G, D, F). The main result of this section is Theorem 2.3, which establishes existence of solutions to the CDMP under these assumptions and a mild condition on F.

Assumption 1 There exists a measurable map $\pi : \mathbb{R}^n \to G$ such that $\pi(y) = y$ for $y \in G$ and $\pi(y) - y \in D(\pi(y))$ for every $y \in \mathbb{R}^n$.

Assumption 2 Consider the SP associated with the data (G, D). Given $\psi_n, \psi \in \mathcal{D}[0, T]$ such that $\psi_n \to \psi$, if (ϕ_n, η_n) solve the SP for ψ_n, $n \in \mathbb{N}$, then there exists

(ϕ, η) such that (along a subsequence) $(\phi_n, \eta_n) \to (\phi, \eta)$ and (ϕ, η) solve the SP for ψ.

Remark 2.1 Assumption 1 reduces to Assumption 3.1 of [9] in the setting of that paper (also, in a more general setting, see Definition 4.1 of [6]). As shown in those papers, this assumption is necessary for existence of solutions to the corresponding SP on $\mathcal{D}[0, T]$. Assumption 2 is equivalent to the statement that the set-valued Skorokhod map (SM) preserves relative compactness and has a closed graph. It was shown in Theorem 1.3 of [20] that the SM has a closed graph if there is no $x \in \partial G$ for which there exists a unit vector d with $\{d, -d\} \subset D(x)$. Under this condition, the paper [6] shows that the SM preserves relative compactness (and thus Assumption 2 holds) when certain oscillation estimates (see Eqs. (2.17) and (2.18) therein) are satisfied. Such estimates are available for broad families of SPs. Indeed, they are easily shown to hold when the associated SM is Lipschitz continuous (see Assumption 4 for a sufficient condition for Lipschitz continuity). In addition, when G is the n-dimensional orthant $\mathbb{R}^n_+$ and $D(x) = \{\sum_{i:x_i=0} c_i d_i : c_i \geq 0, i = 1, \ldots, n\}$, these oscillation estimates were shown in [4] to hold if the constraint matrix (the $n \times n$ matrix with d_i as its ith column) is completely-$\mathcal{S}$ (and in this case Assumption 1 holds as well). Other settings where Assumptions 1 and 2 hold can be found in [6,7].

Our proof of existence relies on the following convergence result, which is standard. For completeness, we have included the proof of this lemma in the Appendix.

Lemma 2.2 *Suppose F is an u.s.c. set-valued function on G, such that each $F(x)$, $x \in G$, is a convex, compact subset of $\mathbb{R}^n$, and for every compact set $E \subset G$, $\cup_{x \in E} F(x)$ is bounded. Let a sequence $(\theta_n)_{n\in\mathbb{N}}$ of absolutely continuous functions and a sequence $(\phi_n)_{n\in\mathbb{N}}$ of functions in $\mathcal{D}[0, T]$ that satisfy $\theta(0) = 0$, $\phi(0) \in G$ and, for every $n \in \mathbb{N}$,*

$$\dot{\theta}_n(t) \in F(\phi_n(t)) \qquad \textit{for a.e. } t \in [0, T],$$

be given. If $\theta_n \to \theta$ in the uniform topology and $\phi_n \to \phi$ in the Skorokhod J_1 topology then

$$\dot{\theta}(t) \in F(\phi(t)) \qquad \textit{for a.e. } t \in [0, T].$$

We now come to the main existence result.

Theorem 2.3 *Suppose that (G, D) satisfies Assumptions 1 and 2, and let F be a uniformly bounded u.s.c. set-valued function on G, such that each $F(x)$, $x \in G$,is a convex, compact subset of $\mathbb{R}^n$. Then for every $u \in \mathcal{D}[0, T]$, there exists at least one $\phi \in \mathcal{M}(G, D, F, u)$.*

Proof Since F is uniformly bounded and has a closed graph, it admits a bounded, measurable selection $\xi \colon G \to \mathbb{R}^n$ (see Corollary 10.3 in the Appendix of [14]). Let π be the function from Assumption 1. Fix $u \in \mathcal{D}[0, T]$ and let $\mathcal{D}_c\,[0, T]$ denote the subspace of piecewise constant functions in $\mathcal{D}[0, T]$ that have a finite number of jumps.

We shall now use an approximating sequence to construct a solution ϕ to the CDMP associated with u. Since $\mathcal{D}_c[0,T]$ is dense (with respect to the uniform topology) in $\mathcal{D}[0,T]$, there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{D}_c[0,T]$ such that $u_n(0) = u(0)$ and $u_n \to u$. For $n \in \mathbb{N}$, define J_n to be the finite set of jump points of u_n, and let $\Pi_n \doteq \{0 = t_0^n < t_1^n \cdots < t_{m_n}^n = T\}$ be a finite partition of $[0,T]$ such that Π_n contains J_n and satisfies $\max_{k=1,\dots,m_n} \Delta t_k^n \leq 1/n$, where $\Delta t_k^n \doteq t_k^n - t_{k-1}^n$. For each $n \in \mathbb{N}$, let ϕ_n, ψ_n and η_n be functions in $\mathcal{D}[0,T]$ that are constant on each interval $[t_{k-1}^n, t_k^n)$ and satisfy $\psi_n(0) = u(0)$, $\phi_n(0) = \pi(u(0))$, $\eta_n(0) = \pi(u(0)) - u(0)$, and

$$\begin{aligned} \psi_n(t_k^n) &\doteq \psi_n(t_{k-1}^n) + u_n(t_k^n) - u_n(t_{k-1}^n) + \xi(\phi_n(t_{k-1}^n))\Delta t_k^n \\ \phi_n(t_k^n) &\doteq \pi\left(\phi_n(t_{k-1}^n) + \psi_n(t_k^n) - \psi_n(t_{k-1}^n)\right) \\ \eta_n(t_k^n) &\doteq \eta_n(t_{k-1}^n) + \phi_n(t_k^n) - \phi_n(t_{k-1}^n) - \psi_n(t_k^n) + \psi_n(t_{k-1}^n) \end{aligned} \tag{2.1}$$

for each $k = 1, \dots, m_n$. It follows immediately from the definitions that (ϕ_n, η_n) solve the SP (G, D) for ψ_n. Now, for $n \in \mathbb{N}$, define

$$\theta_n(t) \doteq \int_0^t \xi(\phi_n(s))\,ds \quad \text{for } t \in [0,T]. \tag{2.2}$$

Then $\theta_n(0) = 0$, θ_n is absolutely continuous and for every $0 \leq s \leq t \leq T$, we have

$$\sup_{n\in\mathbb{N}} |\theta_n(t) - \theta_n(s)| \leq M(t-s), \tag{2.3}$$

where M is a uniform bound on F. Thus, by the Arzèla–Ascoli theorem and the fact that the space of Lipschitz continuous functions with Lipschitz constant bounded by M is a closed subspace of the continuous functions, there exists an absolutely continuous function θ such that $\theta_n \to \theta$ (along a suitable subsequence). Along with the fact that $u_n \to u$ and the easily verified inequality

$$\sup_{t\in[0,T]} |\theta_n(s) - \psi_n(s) + u_n(s)| \leq \frac{M}{n}, \tag{2.4}$$

this shows that $\psi_n \to \psi \doteq \theta + u$. Therefore, by Assumption 2, we can assume without loss of generality (by choosing further subsequences, if necessary) that $\phi_n \to \phi$ and $\eta_n \to \eta$, where (ϕ, η) solve the SP (G, D) for ψ. An application of Lemma 2.2, combined with (2.2), then shows that there exists a bounded, measurable function α on $[0,T]$ such that

$$\theta(t) = \int_0^t \alpha(s)\,ds$$

and $\alpha(s) \in F(\phi(s))$ for a.e. $s \in [0, T]$. The quantities $(u, \theta, \alpha, \eta, \phi)$ above satisfy the conditions of Definition 1.1, thus showing that ϕ solves the CDMP for u. This proves the theorem. □

2.2 Uniqueness of solutions to polyhedral CDMPs

In this section, we consider uniqueness of solutions for the class of polyhedral CDMPs introduced in Sect. 1.4. Under the basic assumptions stated in Sect. 2.2.1 below, the main proof of uniqueness is given in Sect. 2.2.2. With the notation of Sect. 1.4, the constraining mechanism for the polyhedral CDMP dynamical system is then completely described by $(\mathcal{N}_{\text{SP}}, \mathcal{D}_{\text{SP}})$, where

$$\mathcal{N}_{\text{SP}} \doteq \{n_k, k \in \mathbb{K}\} \qquad \text{and} \qquad \mathcal{D}_{\text{SP}} \doteq \{d_k, k \in \mathbb{K}\}.$$

By convention, the case $K = 0$ will correspond to $G = \mathbb{R}^n$ and $\mathcal{N}_{\text{SP}} = \mathcal{D}_{\text{SP}} = \emptyset$ (also referred to as a DMP). On the other hand, when $I = 1$ and $b \equiv 0$, the CDMP reduces to the classical SP (cf. [9]) associated with $(\mathcal{N}_{\text{SP}}, \mathcal{D}_{\text{SP}})$.

The velocity field F of a polyhedral CDMP (G, D, F) is given in (1.10) in terms of $(\mathcal{C}, \mathcal{B}, f^c)$. Throughout Sect. 2.2 we set $f^c = 0$ (for general f^c see Remark 3.6). In this case, for the purpose of studying uniqueness of solutions to the CDMP, we will find it convenient to use an alternative representation of the data $(\mathcal{C}, \mathcal{B})$ describing the interior dynamics. This representation can be given in terms of certain sets $\mathcal{N}_{\text{DI}}$ and $\mathcal{D}_{\text{DI}}$ that play an analogous role in the analysis of the interior dynamics as the sets $\mathcal{N}_{\text{SP}}$ and $\mathcal{D}_{\text{SP}}$ play in the study of the constraining mechanism on the boundary. First, we define a neighbouring relation between the cones $C_i, i \in \mathbb{I}$.

Definition 2.4 For $i, j \in \mathbb{I}$, the cones C_i and $C_j \in \mathcal{C}$ are said to be neighbours (denoted $C_i \sim C_j$) if and only if the dimension of the affine hull of $\overline{C}_i \cap \overline{C}_j$ is $n - 1$.

Define

$$\mathcal{E}(\mathcal{C}) \doteq \{(i, j) \in \mathbb{I}^2 : C_i \sim C_j\} \tag{2.5}$$

to be the set of neighbouring cones in $\mathcal{C}$, and for $(i, j) \in \mathcal{E}(\mathcal{C})$, define ν_{ij} to be the unique unit normal to the affine hull of $\overline{C}_i \cap \overline{C}_j$ that satisfies

$$\langle \nu_{ij}, x \rangle < 0 \quad \text{for } x \in C_i. \tag{2.6}$$

Also, let

$$b_{ij} \doteq b_i - b_j \quad \text{for } (i, j) \in \mathcal{E}(\mathcal{C})$$

and define

$$\mathcal{N}_{\text{DI}} \doteq \{\nu_{ij}, (i, j) \in \mathcal{E}(\mathcal{C})\} \qquad \text{and} \qquad \mathcal{D}_{\text{DI}} \doteq \{b_{ij}, (i, j) \in \mathcal{E}(\mathcal{C})\}.$$

Remark 2.5 Note that one cannot uniquely recover the data $(\mathcal{C}, \mathcal{B})$, and therefore F, from the representation $(\mathcal{N}_{\mathrm{DI}}, \mathcal{D}_{\mathrm{DI}})$ since the latter is invariant under translations of $\mathcal{B}$ by a constant drift b. However, as shown below, questions of uniqueness only depend on the information encoded in $(\mathcal{N}_{\mathrm{DI}}, \mathcal{D}_{\mathrm{DI}})$. Let $\mathcal{N} = (\mathcal{N}_{\mathrm{DI}}, \mathcal{N}_{\mathrm{SP}})$ and $\mathcal{D} = (\mathcal{D}_{\mathrm{DI}}, \mathcal{D}_{\mathrm{SP}})$. We will refer to $(\mathcal{N}, \mathcal{D})$ as a representation for the polyhedral CDMP (G, D, F) whenever (G, D) is defined in terms of $(\mathcal{N}_{\mathrm{SP}}, \mathcal{D}_{\mathrm{SP}})$ and F is obtained from some $(\mathcal{C}, \mathcal{B})$ that admits the representation $(\mathcal{N}_{\mathrm{DI}}, \mathcal{D}_{\mathrm{DI}})$.

2.2.1 Basic assumptions

As mentioned in the introduction, uniqueness results have been established for the case of a single surface of discontinuity under condition (1.6). Assumption 3 below stipulates that a similar condition hold for each hyperplane of discontinuity. Note that this assumption is analogous to the condition (1.11) which we imposed on the data associated with the constraining mechanism.

Assumption 3 Given $(\mathcal{N}_{\mathrm{DI}}, \mathcal{D}_{\mathrm{DI}})$, for every $(i, j) \in \mathcal{E}(\mathcal{C})$, either $\langle \nu_{ij}, b_{ij} \rangle > 0$ or $b_{ij} = 0$.

We now present the main condition for uniqueness. It is analogous to Assumption 2.1 of [9]. A set $B \subset \mathbb{R}^n$ is said to be symmetric if $B = -B$.

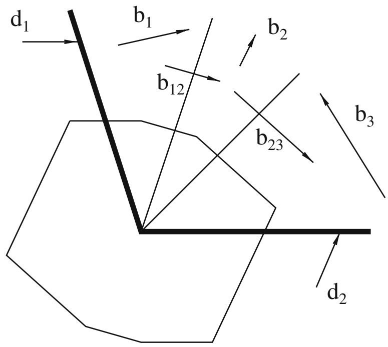
Assumption 4 (Existence of a Lyapunov set) Given the representation $(\mathcal{N}, \mathcal{D})$ of the problem data, there exists a compact, symmetric, convex set $B \subset \mathbb{R}^n$ with $0 \in B^o$ such that if $\vartheta(z)$ is the set of inward normals to B at $z \in \partial B$, then the following properties hold.

(i) If $z \in \partial B$ and $|\langle z, \nu_{ij} \rangle| < 1$ for some $(i, j) \in \mathcal{E}(\mathcal{C})$, then $\langle b_{ij}, \vartheta \rangle = 0$ for all $\vartheta \in \vartheta(z)$.
(ii) If $z \in \partial B$ and $|\langle z, n_k \rangle| < 1$ for some $k \in \mathbb{K}$, then $\langle d_k, \vartheta \rangle = 0$ for all $\vartheta \in \vartheta(z)$.

Remark 2.6 (i) Assumption 4 is an analogue of a condition of Dupuis and Ishii introduced in [9], which guarantees uniqueness of solutions to the SP on polyhedral domains. It is worthwhile to note that although the CDMP can in a sense be viewed as a mixture of a DMP and an SP, uniqueness of solutions to the CDMP cannot be studied simply by breaking it down into these constituent parts (cf. Example 2.12).
(ii) A convex duality method for the construction of such Lyapunov sets was introduced in [11] in the context of polyhedral SPs, and then applied to establish regularity of the corresponding SM for several concrete classes of SPs in [12,13]. This method is likely to also be useful for verifying Assumption 4 for polyhedral CDMPs and will be explored in future work.

Remark 2.7 The Lyapunov set condition of Assumption 4 can be visualised in the planar case, $n = 2$, in a simple way (see Fig. 2). In the two-dimensional setting, the geometry of the problem can be associated with a set of rays emanating from the origin, defining the boundary faces ∂G_i of the domain G and the regions C_i. It is

Fig. 2 A two-dimensional CDMP and a corresponding Lyapunov set



not hard to see that Assumption 4 is then satisfied if one can find a symmetric convex polyhedron with the following property: every ray defining the boundary between two regions C_i and C_j (respectively, the boundary face ∂G_i) intersects properly a side of the polyhedron that is parallel to the corresponding direction b_{ij} (respectively, d_i).

The following consequence of Assumption 4 is proved in a manner similar to Lemma 2.1 of [9]. We provide the proof in the Appendix for the sake of completeness.

Lemma 2.8 *Let Assumptions 3 and 4 hold. Let B be as in Assumption 4, $z \in \partial B$ and $\vartheta \in \vartheta(z)$. Then*

$$\langle z, n_k \rangle \langle \vartheta, d_k \rangle \leq 0 \quad \textit{for } k \in \mathbb{K} \tag{2.7}$$

and

$$\langle z, \nu_{ij} \rangle \langle \vartheta, b_{ij} \rangle \leq 0 \quad \textit{for } (i, j) \in \mathcal{E}(\mathcal{C}). \tag{2.8}$$

2.2.2 Proof of uniqueness

Theorem 2.9 below provides a general sufficient condition, namely the validity of Assumptions 3 and 4, for uniqueness of solutions and Lipschitz continuity of the solution map. Combining the uniqueness and existence results, Theorem 2.11 shows that if the polyhedral CDMP satisfies Assumptions 1, 2, 3 and 4, then the solution mapping is well defined and Lipschitz continuous on all of $\mathcal{D}[0, T]$. For $f \in \mathcal{D}[0, T]$, we will write $\sup_{s\in[0,t]} |f(s)|$ as $||f||_t$.

Theorem 2.9 *Suppose that the polyhedral CDMP (G, D, F) admits a representation $(\mathcal{N}, \mathcal{D})$ that satisfies Assumptions 3 and 4. Let $u_1, u_2 \in \mathcal{D}[0, T]$ and suppose that for $i = 1, 2$, $\phi_i \in \mathcal{M}(G, D, F, u_i)$. Then there exists a constant κ, which is independent of T, u_i and ϕ_i, $i = 1, 2$, such that*

$$||\phi_1 - \phi_2||_T \leq \kappa ||u_1 - u_2||_T.$$

The proof of Theorem 2.9 relies on the following result.

Lemma 2.10 *For each $i, j \in \mathbb{I}$, $i \neq j$, and $x \in \overline{C}_i, y \in \overline{C}_j$, there exist $m \in \{1, \dots, I-1\}$ and $i_0, \dots, i_m \in \mathbb{I}$ such that $i_0 = i, i_m = j$, $(i_l, i_{l+1}) \in \mathcal{E}(\mathcal{C})$ and $\langle x - y, \nu_{i_l i_{l+1}} \rangle \leq 0$ for all $l = 0, \dots, m-1$. We shall denote the collection of such ordered sets $(i_0, \dots, i_m)$ by $S(i, j, x, y)$.*

Proof Fix $i, j \in \mathbb{I}$, $i \neq j$, and define

$$\mathcal{U}_{ij} \doteq \left\{ (x, y) \in \overline{C}_i \times \overline{C}_j : \begin{array}{l} \text{there exist } m \in \{1, \dots, I-1\} \text{ and} \\ i_0, \dots, i_m \in \mathbb{I} \text{ such that } i_0 = i, i_m = j, \\ \text{and for } l = 0, \dots, m-1, (i_l, i_{l+1}) \in \mathcal{E}(\mathcal{C}) \\ \text{and } \langle \nu_{i_l i_{l+1}}, x - y \rangle \leq 0 \end{array} \right\}.$$

In order to prove the lemma, we need to show that $\mathcal{U}_{ij} = \overline{C}_i \times \overline{C}_j$. We will start by showing that $\mathcal{U}_{ij}$ contains a dense subset (S_{ij} defined below) of $\overline{C}_i \times \overline{C}_j$. Let

$$S_0 \doteq \left\{ x \in \mathbb{R}^n : \begin{array}{l} \langle x, \nu_{q'r'} \rangle = 0, \ \langle x, \nu_{qr} \rangle = 0, \quad \text{for some} \\ (q', r'), (q, r) \in \mathcal{E}(\mathcal{C}) \text{ with } (q, r) \neq (q', r') \end{array} \right\}$$

be the set of points that lie at the intersection of at least two hyperplanes of discontinuity of the velocity field, and for $(x, y) \in C_i \times C_j$, let $\ell(x, y) \doteq \{\alpha x + (1-\alpha) y : \alpha \in [0, 1]\}$ be the line segment joining x and y. Then define

$$S_{ij} \doteq \left\{ (x, y) \in C_i \times C_j : \ell(x, y) \cap S_0 = \emptyset \right\}.$$

We first claim that $S_{ij} \subseteq \mathcal{U}_{ij}$. To see why this is true, let $(x, y) \in S_{ij}$ be given, and define $x_t \doteq (1-t)x + ty$ for $t \in [0, 1]$. Due to the convexity of the domain and the fact that $x, y \in G^\circ$, we have $x_t \in G^\circ$ for $t \in [0, 1]$. Let $t_1 \doteq \inf\{t \geq 0 : x_t \notin C_i\}$. Clearly $t_1 \in (0, 1)$ and, since $x_{t_1} \in \partial C_i \cap G^\circ$, we must have $\langle \nu_{ik}, x_{t_1} \rangle = 0$ for some $k \in \mathbb{K}$ such that $C_k \sim C_i$. In fact, since $x_{t_1} \notin S_0$ (because $x_{t_1} \in \ell(x, y)$ and $(x, y) \in S_{ij}$), there is exactly one such k. This implies that x_{t_1} lies in the relative interior of $\partial C_i \cap \partial C_k$ and $\langle x_t, \nu_{ik} \rangle$ changes sign at $t = t_1$. In turn, this means that there exists $\varepsilon_0 \in (0, 1 - t_1)$ such that $x_{t_1 + t} \in C_k$ for all $t \in [0, \varepsilon_0]$ and, since $\langle x, \nu_{ik} \rangle < 0$ due to (2.6) and $y = x_1$, such that

$$\langle \nu_{ik}, x - y \rangle < 0.$$

If $k = j$ then this shows that $(x, y) \in \mathcal{U}_{ij}$, with $m = 1$ in the definition of $\mathcal{U}_{ij}$. If $k \neq j$ then set $i_1 \doteq k$, $x^1 \doteq x_{t_1 + \varepsilon_0}$ and note that $x^1 \in C_{i_1}$ and $(x^1, y) \in S_{i_1 j}$ (since $\ell(x^1, y) \subset \ell(x, y)$ and $(x, y) \in S_{ij}$). The above argument can thus be iterated (replacing x and i by x^1 and i_1, respectively) so as to obtain $m \in \{1, \dots, I-1\}$ and a sequence $i = i_0, i_1, \dots, i_{m-1}, i_m = j$ such that C_{i_l} is the lth cone intersected by x_t (ordered chronologically as t grows from 0 to 1). It follows that $(x, y) \in \mathcal{U}_{ij}$ and $\{i_0, \dots, i_m\} \in S(i, j, x, y)$. This proves the claim. In particular, since S_{ij} is dense in $\overline{C}_i \times \overline{C}_j$, we conclude that $\mathcal{U}_{ij}$ is dense in $\overline{C}_i \times \overline{C}_j$.

Now choose an arbitrary $(x, y) \in \overline{C}_i \times \overline{C}_j$, let $(x_n, y_n) \in \mathcal{U}_{ij}$ be such that $x_n \to x$, $y_n \to y$ and let $(i_{0,n}, \dots, i_{m_n,n}) \in S(i, j, x_n, y_n)$. Since $\mathbb{I}$ is a finite set and $i_{j,n} \neq i_{k,n}$

for $j \neq k$, $n \in \mathbb{N}$, there exists a fixed index set $S = \{i_0, \ldots, i_m\}$ such that $S \in S(i, j, x_{n_k}, y_{n_k})$ along some subsequence $\{n_k\}$. Then for all $l = 0, 1, \ldots, m-1$, we have $\langle x_{n_k} - y_{n_k}, \nu_{i_l, i_{l+1}} \rangle \leq 0$, $k \in \mathbb{N}$, and therefore on taking the limit as $k \to \infty$, we have $\langle x - y, \nu_{i_l, i_{l+1}} \rangle \leq 0$. Thus $(x, y) \in \mathcal{U}_{ij}$ with $S \in S(i, j, x, y)$ and the result follows. $\square$

Proof of Theorem 2.9 For $i = 1, 2$, let $\alpha_i, \gamma_i \in \mathcal{BM}[0, T]$, $\theta_i \in \mathcal{AC}[0, T]$ and $\eta_i \in \mathcal{D}[0, T]$ be such that (i) through (v) of Definition 1.1 are satisfied with $(u, \phi, \theta, \alpha, \eta, \gamma)$ there replaced by $(u_i, \phi_i, \theta_i, \alpha_i, \eta_i, \gamma_i)$. Let $\zeta_i \in \mathcal{D}[0, T]$ be defined as

$$\zeta_i(t) \doteq \theta_i(t) + \eta_i(t), \quad t \in [0, T].$$

Let $c \doteq \sup_{0 \leq t \leq T} |u_1(t) - u_2(t)|$. We will prove that

$$\zeta_1(t) - \zeta_2(t) \in cB \quad \text{for all } t \in [0, T]. \tag{2.9}$$

Since B is compact, this will clearly prove the result. We will argue by contradiction. Suppose there exists $a \in (c, \infty)$ such that $\zeta_1(t) - \zeta_2(t) \notin aB^o$ for some $t \in [0, T]$. Let

$$\tau \doteq \inf\{t \in [0, T] : \zeta_1(t) - \zeta_2(t) \notin aB^o\}.$$

Following [9], we divide the analysis into two cases.

Case 1 $\zeta_1(\tau-) - \zeta_2(\tau-) \in \partial(aB)$.

Let $z \doteq \zeta_1(\tau-) - \zeta_2(\tau-)$. Let $\vartheta \in \vartheta(z/a)$. Then for all $t \in [0, \tau)$, since $\zeta_1(t) - \zeta_2(t) \in aB^\circ$, it follows that

$$\langle z - \zeta_1(t) + \zeta_2(t), \vartheta \rangle < 0.$$

Note that $z - \zeta_1(t) + \zeta_2(t)$ can be expressed as

$$\int_{(t,\tau)} (\alpha_1(s) - \alpha_2(s))\, ds + \int_{(t,\tau)} (\gamma_1(s) d|\eta_1|(s) - \gamma_2(s)\, d|\eta_2|(s)).$$

Thus at least one of the following three properties must hold along a sequence $\{t_n\}$, $t_n \uparrow \tau$.

1. $\displaystyle\int_{(t_n,\tau)} \langle \gamma_1(s), \vartheta \rangle\, d|\eta_1|(s) < 0;$
2. $\displaystyle\int_{(t_n,\tau)} \langle \gamma_2(s), \vartheta \rangle\, d|\eta_2|(s) > 0;$
3. $\displaystyle\int_{(t_n,\tau)} \langle \alpha_1(s) - \alpha_2(s), \vartheta \rangle\, ds < 0.$

Statements 1 and 2 above lead to a contradiction exactly as in the proof of Theorem 2.2 of [9], and so they cannot hold.

Now suppose that statement 3 holds. Following Remark 1.2, for $i = 1, 2$, we can find $\lambda_{i,k} \in \mathcal{BM}([0, T] : [0, 1])$, $k \in \mathbb{I}$, such that for a.e. $t \in [0, T]$, $\sum_{k\in\mathbb{I}} \lambda_{i,k}(t) = 1$, $\lambda_{i,k}(t) 1\!\!1_{\{\phi_i(t)\notin\overline{C}_k\}} = 0$ and $\alpha_i(t) = \sum_{k\in\mathbb{I}} \lambda_{i,k}(t) b_k$. Setting $\lambda_{jk}(t) \doteq \lambda_{1,j}(t)\lambda_{2,k}(t)$, we see that for $s \in [0, T]$,

$$\alpha_1(s) - \alpha_2(s) = \sum_{j\in\mathbb{I}} \sum_{k\in\mathbb{I}} \left(b_j - b_k\right) \lambda_{jk}(s) = \sum_{j\in\mathbb{I}} \sum_{k\in\mathbb{I}} b_{jk} \lambda_{jk}(s).$$

This, along with statement 3 above, guarantees the existence of $(j, k) \in \mathbb{I} \times \mathbb{I}$ such that

$$\langle b_{jk}, \vartheta\rangle < 0 \quad \text{and} \quad \int_{(t_n,\tau)} \lambda_{jk}(s)\, ds > 0 \qquad \text{for } n \in \mathbb{N} \tag{2.10}$$

(with the latter inequality holding along a further subsequence of $\{t_n\}$, if needed). Taking the limit as $n \to \infty$ in the above display and using the fact that $\lambda_{jk}(t) 1\!\!1_{\{\phi_1(t)\notin\overline{C}_j\}} 1\!\!1_{\{\phi_2(t)\notin\overline{C}_k\}} = 0$ we see that $\phi_1(\tau-) \in \overline{C}_j$ and $\phi_2(\tau-) \in \overline{C}_k$. Then, by Lemma 2.10, there exists a finite sequence $\{i_0, \ldots, i_m\} \in S(j, k, \phi_1(\tau-), \phi_2(\tau-))$ that satisfies the relations

$$\langle \phi_1(\tau-) - \phi_2(\tau-), \nu_{i_l i_{l+1}}\rangle \le 0 \tag{2.11}$$

for every $l \in \{0, \ldots, m-1\}$. Also, since $b_{jk} = b_j - b_k = \sum_{l=0}^{m-1}(b_{i_l} - b_{i_{l+1}}) = \sum_{l=0}^{m-1} b_{i_l i_{l+1}}$, the first relation in (2.10) yields

$$\langle b_{i_l,i_{l+1}}, \vartheta\rangle < 0$$

for some $l \in \{0, 1, \ldots, m-1\}$. Due to Assumption 4 and Lemma 2.8, the last display implies that $\langle z/a, \nu_{i_l i_{l+1}}\rangle \ge 1$. Hence we now have

$$\begin{aligned} a \le \langle z, \nu_{i_l i_{l+1}}\rangle &= \langle \phi_1(\tau-) - \phi_2(\tau-), \nu_{i_l i_{l+1}}\rangle - \langle u_1(\tau-) - u_2(\tau-), \nu_{i_l i_{l+1}}\rangle \\ &\le \langle \phi_1(\tau-) - \phi_2(\tau-), \nu_{i_l i_{l+1}}\rangle + c, \end{aligned}$$

which implies that $\langle \phi_1(\tau-) - \phi_2(\tau-), \nu_{i_l i_{l+1}}\rangle > 0$. This contradicts (2.11) and so statement 3 also cannot hold.

Case 2 $\zeta_1(\tau-) - \zeta_2(\tau-) \in aB^o$. This time set $z \doteq \zeta_1(\tau) - \zeta_2(\tau)$ and let $b \in [a, \infty)$ be such that $z \in \partial(bB)$. Let $\vartheta \in \vartheta(z/b)$. Then by the convexity of B, $\langle z - \zeta_1(\tau-) + \zeta_2(\tau-), \vartheta\rangle < 0$. Noting that

$$z - \zeta_1(\tau-) + \zeta_2(\tau-) = \int_{\{\tau\}} (\gamma_1(s)\, d|\eta_1|(s) - \gamma_2(s)\, d|\eta_2|(s)),$$

we are led to a contradiction exactly as in Theorem 2.2 of [9]. This proves (2.9) and hence the result.

□

An immediate consequence of Theorems 2.9 and 2.3 is the following.

Theorem 2.11 *Suppose that the polyhedral CDMP* (G, D, F) *admits the representation* $(\mathcal{N}, \mathcal{D})$*, where* $(\mathcal{N}_{SP}, \mathcal{D}_{SP})$ *satisfy Assumptions 1 and 2. Then for every* $T > 0$ *and* $u \in \mathcal{D}[0, T]$*, there exists at least one* $\phi \in \mathcal{M}(G, D, F, u)$*. Furthermore, if* $(\mathcal{N}_{DI}, \mathcal{D}_{DI})$ *also satisfy Assumptions 3 and 4, then the solution* ϕ *is unique. In this case, we use* Φ *to denote the solution map and write* $\phi = \Phi(u)$*. Finally, for some* $\kappa \in (0, \infty)$*,*

$$||\Phi(u_1) - \Phi(u_2)||_T \leq \kappa||u_1 - u_2||_T,$$

for all $T > 0$ *and* $u_1, u_2 \in \mathcal{D}[0, T]$.

2.3 Examples

We now provide examples of CDMPs in order to illustrate the applicability of Theorem 2.11. We begin with an example that shows that the question of uniqueness of solutions to a CDMP cannot be decoupled by examining uniqueness for the associated SP and a naturally associated DMP separately. This emphasises the need for a common framework that considers both these problems. Define d on the boundary ∂G of G as follows: for $x \in \partial G$,

$$d(x) \doteq D(x) \cap S^1(0), \tag{2.12}$$

where $S^1(0)$ is the unit sphere in $\mathbb{R}^n$.

Example 2.12 Consider a CDMP that has domain $\mathbb{R}^2_+$ and boundary constraint data given by

$$n_1 = (1, 0) \qquad d_1 = (1, -1) \quad n_2 = (0, 1) \qquad d_2 = (0, 1),$$

and has the interior of G partitioned into the cones $C_1 = \{x \in G^o : \langle x, \nu_{12}\rangle < 0\}$ and $C_2 = \{x \in G^o : \langle x, \nu_{12}\rangle > 0\}$, whose common boundary lies in the hyperplane normal to $\nu_{12} = \{1, -1\}$, with drift b_i in the cone C_i, for $i = 1, 2$, defined by

$$b_1 = (2, 0) \qquad b_2 = (0, 0)\,.$$

Also, as usual, let $b_{12} = b_1 - b_2 = (2, 0)$ and let F be the set-valued function obtained by the convexification of the vector field associated with (b_1, b_2) and (C_1, C_2). Then it is easy to check that $\langle b_{12}, \nu_{12}\rangle = 2$ and $\langle d_i, n_i\rangle = 1$ for $i = 1, 2$, and so Assumption 3 is satisfied for the CDMP data. In particular, by the comment following Eq. (1.6), this implies that solutions to the associated DMP (extended to $\mathbb{R}^2$ in the obvious manner) are unique. Moreover, it is straightforward to check that the set $B = \{x \in \mathbb{R}^2 : |x_1| \leq 1\} \cap \{x \in \mathbb{R}^2 : |x_1 + x_2| \leq 2\}$ satisfies Assumption 4 for the data $\{(n_1, n_2), (d_1, d_2)\}$ related to the SP component of the CDMP. In addition, the SP data is also easily seen to satisfy Assumptions 1 and 2. Thus, invoking Theorem 2.11, we see that the corresponding SM is Lipschitz continuous on $\mathcal{D}[0, T]$ and solutions to the CDMP exist for all input functions $u \in \mathcal{D}[0, T]$.

However, it is easy to see that there exists no set that satisfies Assumption 4 for the data $\{(\nu_{12}, b_{12}), (n_1, d_1), (n_2, d_2)\}$. In fact, there exist input functions u for which the CDMP admits multiple solutions. Indeed, consider the function

$$u(t) \doteq (-1, 1)\, t \quad \text{for } t \in [0, 1].$$

Then the function $\phi^{(1)}(t) \doteq 0$ for $t \in [0, 1]$ has the form $\phi^{(1)}(t) = u(t) + \theta^{(1)}(t) + \eta^{(1)}(t)$, where $\theta = 0$ and $\eta = -u = d_1 t$. Thus $d\theta^{(1)}(t)/dt = 0 \in F(0) = F(\phi^{(1)})(t)$, $d\eta^{(1)}(t)/dt \in d(0) = d(\phi^{(1)}(t))$ for all $t \in [0, 1]$, and $\phi^{(1)}$ defines one solution to the CDMP associated with u. On the other hand, the function $\phi^{(2)}(t) \doteq (1, 1)t$ for $t \in [0, \infty)$ has the representation $\phi^{(2)}(t) = u(t) + \theta^{(2)}(t) + \eta^{(2)}(t)$, where $\theta^{(2)}(t) = b_1 t$ and $\eta^{(2)}(t) = 0$. Thus $d\theta^{(2)}/dt = b_1 \in F(\phi^{(2)}(t))$ and $d\eta^{(2)}/dt = 0 \in d(\phi^{(2)}(t))$ for all t, and so $\phi^{(2)} \neq \phi^{(1)}$ defines another solution to the CDMP associated with u.

Next, we describe a simple DMP for which Assumption 4 does not hold and which exhibits multiple solutions.

Example 2.13 The DMP, depicted in Fig. 3a, has the following data.

$$\begin{aligned}
C_1 &= \{x \in \mathbb{R}^2 : x_1 < 0\}, \quad C_2 = \{x \in \mathbb{R}^2 : x_1 > 0, x_1 - x_2 < 0\},\\
C_3 &= \{x \in \mathbb{R}^2 : x_1 > 0, x_1 - x_2 > 0\},\\
b_1 &= (1, 0), \quad b_2 = (-1, 0), \quad b_3 = (-3, -1).
\end{aligned}$$

One computes $b_{12} = (2, 0)$, $b_{23} = (2, 1)$ and $b_{31} = (-4, -1)$, and $\nu_{12} = (1, 0)$, $\nu_{23} = (1, -1)$ and $\nu_{31} = (-1, 0)$. The discussion in Remark 2.7 suggests that there exists no set satisfying Assumption 4 since, given ν_{12} and ν_{23}, the directions b_{12} and b_{23} only allow for a nonconvex set. To make this argument precise, assume such a set, denoted by B, exists, and let y (resp., z) be the point where the boundary ∂B intersects the ray $\{x : x_1 = x_2 > 0\}$ that is perpendicular to ν_{23} (resp., the ray $\{x : x_1 = 0, x_2 > 0\}$ perpendicular to ν_{12}). By Assumption 4(ii), it is easy to see that the vector $a = (-1, 2)$ that is perpendicular to b_{23} (resp., the vector $b = (0, 1)$ perpendicular to b_{12}) is an outward normal to ∂B at y (resp., at z). Note that the ray $\{y + sa : s > 0\}$ necessarily intersects the ray $\{z + tb : t > -z_2\}$. This can occur for $t > 0$, in which case we have two outward normal rays to ∂B intersecting outside B, or the intersection can occur for $-z_2 < t \leq 0$, in which case it follows that an outward normal ray at y intersects B at some point other than y. In both cases, the convexity of B is violated.

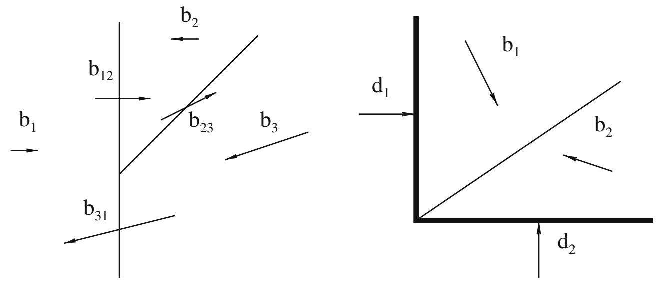


Fig. 3 **a** DMP of Example 2.13. **b** CDMP of Example 2.14(i)

We now show that, in fact, multiple solutions exist to the corresponding DI. Indeed, with

$$u(t) = \left(0, \frac{1}{4}\right) t,$$

one has

$$\phi^{(1)}(t) \doteq \left(0, \frac{1}{4}\right) t = \frac{1}{2} b_1 t + \frac{1}{2} b_2 t + u(t)$$
$$\phi^{(2)}(t) \doteq 0 = \frac{3}{4} b_1 t + \frac{1}{4} b_3 t + u(t),$$

and clearly $\frac{1}{2}b_1 + \frac{1}{2}b_2 \in F(\phi^{(1)}(t))$ and $\frac{3}{4}b_1 + \frac{1}{4}b_3 \in F(\phi^{(2)}(t))$ for all $t \geq 0$. Thus both $\phi^{(1)}$ and $\phi^{(2)}$ solve the DMP for u.

We now discuss a case that appears in queueing applications.

Example 2.14 This model arises in functional law of large numbers and central limit theorem limits to a queueing system with n customer classes and a single server using the so-called "Weighted serve-the-longer-queue" (weighted SLQ) service discipline. In this discipline, customers of class i are queued in buffer i ($i = 1, \ldots, n$) and, for every t, the server serves class-i customers whenever $\alpha_i Q_i(t) > \alpha_j Q_j(t)$ for all $j \neq i$; if the set $\text{argmax}_i \alpha_i Q_i(t)$ has more than one element, the class with the largest index is served. Here, $\alpha_i > 0$ are fixed, and $Q_i(t)$ denotes the number of class-i customers in the system at time t. Moreover, μ represents the mean service rate of a customer and λ_i denotes the (long-run average) arrival rate of class i customers.
(i) When there are two classes, the polyhedral CDMP associated with this model has domain $G = \mathbb{R}_+^2$, with constraint data $n_1 = d_1 = e_1$, $n_2 = d_2 = e_2$ and the interior divided into the two regions

$$C_1 = \{x \in G^o : \alpha_1 x_1 < \alpha_2 x_2\}, \quad C_2 = \{x \in G^o : \alpha_1 x_1 > \alpha_2 x_2\},$$

with corresponding drifts $b_1 = (\lambda_1, \lambda_2 - \mu)$ and $b_2 = (\lambda_1 - \mu, \lambda_2)$ (see Figure 3(b)). Noting that $b_{12} = (\mu, -\mu)$, it is easy to check directly that the set defined by

$$B = \left\{ (x_1, x_2) : |x_1 + x_2| \leq \alpha_1 + \alpha_2, |x_1| \leq \frac{1}{2}\alpha_1 + \alpha_2, |x_2| \leq \alpha_1 + \frac{1}{2}\alpha_2 \right\},$$

satisfies Assumption 4. Moreover, since the constraint vector field D is the normal vector field in this case, Assumptions 1 and 3 hold trivially, while Assumption 2 holds by Remark 2.1, since $D(x)$ does not contain a line for any x, and the constraint matrix is completely-$\mathcal{S}$. By Theorem 2.11, the associated solution map is therefore continuous on $\mathcal{D}[0, T]$.

In order to study the fluid limit, it suffices to consider the CDMP with input $u = 0$, in which case the boundary constraint directions are only relevant at the origin. On the other hand, for the diffusion limit, the input trajectories u to the CDMP represent diffusion paths and so non-trivial constraining action takes place throughout the boundary of the non-negative orthant (note, however, that we do not prove here that these processes correspond to the respective scaling limits).

(ii) We shall now investigate the uniqueness of the CDMP associated with the equally weighted SLQ model with three classes, for inputs $u = 0$. As mentioned above, this is sufficient for the characterisation of the functional law of large numbers limits of such systems. The CDMP described in (i), with $u = 0$, has the property that no constraint action takes place on the boundary of the orthant, except at the origin. As a result, the same solution can be obtained with a domain G that is larger than the orthant, and this can be used to simplify the corresponding polyhedral CDMP. An analogous statement holds for the model with three classes under consideration, and the corresponding simplified CDMP is as follows. Denoting $e = (1, 1, 1)/\sqrt{3}$, the simplified CDMP has domain $G = \{x : \langle x, e\rangle \geq 0\}$, regions

$$C_i = \{x \in G^o : x_i > x_j, \text{ all } j \neq i\}, \quad i = 1, 2, 3,$$

and data $d_1 = n_1 = e$, $b_{ij} = \mu(e_j - e_i)$, $\nu_{ij} = e_j - e_i$, for $i, j = 1, 2, 3, i \neq j$. Assumption 3, namely $\langle \nu_{ij}, b_{ij}\rangle > 0$, holds. We now show that the following set satisfies Assumption 4:

$$B = \left\{ x \in \mathbb{R}^3 : |\langle x, e\rangle| \leq 1, \ |\langle x, e_i + e_j - 2e_k\rangle| \leq 3 \text{ for all } i, j, k \text{ distinct} \right\}.$$

We first verify part (i) of Assumption 4. Letting $z \in \partial B$ be such that

$$|\langle z, \nu_{12}\rangle| < 1, \tag{2.13}$$

we will show that $\langle b_{12}, \vartheta\rangle = 0$ for every normal ϑ to ∂B at z. First, note that the equality $z_1 + z_3 - 2z_2 = 3$ does not hold. Indeed, if $z_1 + z_3 - 2z_2 = 3$ then, using (2.13), $z_1 + z_2 - 2z_3 = 3(z_1 - z_2) - 2(z_1 + z_3 - 2z_2) = 3(z_1 - z_2) - 6 < -3$, violating the inequality $|z_1 + z_2 - 2z_3| \leq 3$ that must hold by the definition of B. A similar calculation shows that the equalities $|z_1 + z_3 - 2z_2| = 3$ and $|z_2 + z_3 - 2z_1| = 3$ do not hold. As a result, any normal ϑ to ∂B at z is a linear combination $e_1 + e_2 - 2e_3$ and e. This immediately ensures that $\langle \vartheta, b_{12} \rangle = 0$. By symmetry, the cases with ν_{12} in (2.13) replaced by ν_{13} and ν_{23}, respectively, can be treated similarly. This shows that part (i) of Assumption 4 holds.

To verify part (ii) of the assumption, recall that $n_1 = d_1 = e$ and let $z \in \partial B$ be such that $|\langle z, e \rangle| < 1$. Then any normal ϑ to ∂B at z is a linear combination of $e_i + e_j - 2e_k$, (i, j, k) distinct. Hence $\langle \vartheta, e \rangle = 0$.

Existence and uniqueness of the solution to the CDMP with $u = 0$ follow.

We conclude with a simple multi-dimensional example that demonstrates that our method yields new results even in the pure-DMP setting.

Example 2.15 We consider a DMP in $\mathbb{R}^n$ in which the b_i are co-linear. Let $A_1 \subset \cdots \subset A_k \subset A_{k+1} = \mathbb{R}^n$ be closed convex polyhedral cones such that the vector e_1 is an element of the interior of A_1. Let the drift vector field b take the constant value b_i on the interior of each $\tilde{C}_i$, where $\tilde{C}_1 = A_1$, and $\tilde{C}_i = A_i \setminus A_{i-1}$ for $i = 2, \ldots, k+1$. Assume $b_i = r_i e_1$ where

$$r_1 < r_2 < \cdots < r_{k+1} \tag{2.14}$$

are given real numbers (see Fig. 4a for a two-dimensional example). We will show that this data corresponds to a polyhedral DMP for which Assumptions 3 and 4 hold. Although the setting is quite simple, continuity (or even uniqueness) results of the form of Theorem 2.9 do not follow from existing results in the literature. In particular, as already mentioned in Sect. 1.2.2, the uniqueness results of [15] do not cover the current setting. Moreover, neither do the results of [5] since the drift vector field $F(\cdot)$ associated with this DMP is not the negative of a maximal monotone operator. Indeed, consider, for example, the case $A_1 = \{x \in \mathbb{R}^2 : x_1 \geq |x_2|\}$, $A_2 = \mathbb{R}^2$, $b_1 = 0$, $b_2 = e_1$. Then, for $x = e_1 \in \tilde{C}_1^o$ and $y = 2e_1 + 3e_2 \in \tilde{C}_2^o$, we have $\langle -F(x)+F(y), x-y \rangle = \langle -b_1+b_2, -e_1-3e_2 \rangle < 0$, which violates the monotonicity assumption (2.3) of [5].

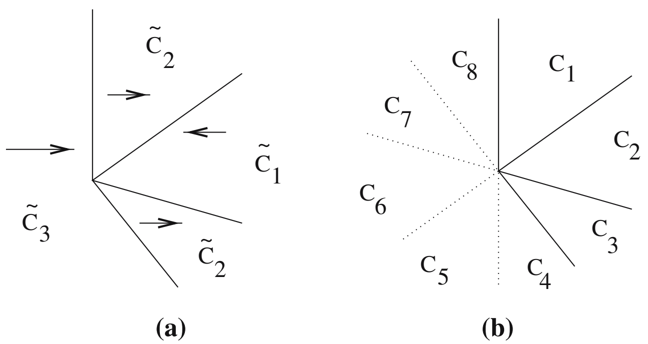


Fig. 4 **a** A DMP with a co-linear velocity vector field - a two-dimensional case of Example 2.15. **b** $\tilde{C}_2$ divided into two convex cones and $\tilde{C}_3$ into five (for simplicity, we use here 1–8 in place of elements of $\mathbb{I}$ for the labels I of C_I)

Since $\tilde{C}_i$ need not all be convex, they do not constitute the part $\mathcal{C}$ (a collection of convex polyhedral cones (1.7)) of the data of a polyhedral DMP. However, we can easily recast the problem above in terms of a polyhedral DMP with convex data $\mathcal{C}$ by splitting $\tilde{C}_i$ into convex polyhedral cones, and assigning the same drift b_i to all convex polyhedral cones that lie within $\tilde{C}_i$ (see Fig. 4b for the convex realization of the DMP in Fig. 4a). More precisely, a generic element of $\mathbb{I}$ will be denoted by I or J, and the sets $\mathbb{I}$ and $\mathcal{C} = \{C_I : I \in \mathbb{I}\}$ will be defined below. Write $A_i = \cap_l G_{il}$, where G_{il} are closed half spaces, relabel G_{il} as G_m, $m = 1, \dots, M$, and consider the 2^M open polyhedral cones (some of which may be empty) $C_I = \cap_{m=1}^M \tilde{G}_{m,n_m}$, indexed by $I = (n_1, \dots, n_M) \in \{0, 1\}^M$, where $\tilde{G}_{m,0} = G_m^o$ and $G_{m,1} = G_m^c$. Let $\mathbb{I}$ be the set of $I \in \{0, 1\}^M$ for which C_I is not empty. Then C_I, $I \in \mathbb{I}$, are disjoint, open polyhedral cones satisfying $\overline{\cup C_I} = \mathbb{R}^n$, and it is not hard to see that each C_I is contained in one of the $\tilde{C}_i$. As a result, the closure of each $\tilde{C}_i$ can be written as the closure of the union of C_I over $I \in \mathbb{I}$ such that $C_I \subset \tilde{C}_i$. The DMP is well-defined by setting $b_I = b_i$ whenever $C_I \subset \tilde{C}_i$.

To verify Assumption 3, note that if C_I and C_J are subsets of the same $\tilde{C}_i$ then by construction $b_I = b_J$, and so the pair I, J satisfies the assumption. If $C_I \subset \tilde{C}_i$ and $C_J \subset \tilde{C}_j$ for $i > j$ then $\langle \nu_{IJ}, e_1 \rangle > 0$ because of the nested construction of the A_i's and because $e_1 \in A_1^o$. Hence due to (2.14), $\langle \nu_{IJ}, b_{IJ} \rangle > 0$ and again the pair I, J satisfies Assumption 3.

We now turn to Assumption 4. Denoting by B the cylinder $\{x \in \mathbb{R}^n : |x_1| \leq 1, x_2^2 + \dots + x_n^2 \leq \epsilon\}$, we claim that the set $\delta^{-1} B$ satisfies our Assumption 4, provided that ϵ and δ are sufficiently small. To see this, note that by construction, for each i, l, ∂G_{il} does not contain e_1. As a result, neither does the affine hull of $\overline{C_I} \cap \overline{C_J}$ for each $I, J \in \mathbb{I}$. Therefore if $z \in \partial B$ and $|\langle z, \nu_{IJ} \rangle| < \delta$ then the distance of z from the set $\{-e_1, e_1\}$ is bounded away from zero uniformly in z, as $\delta \to 0$. We can deduce that if ϵ and δ are sufficiently small then $|\langle z, e_1 \rangle| < 1$ and thus the normal to ∂B at z is orthogonal to e_1, and hence to b_{IJ}.

Although we have exhibited in Examples 2.14 (ii) and 2.15 that one can sometimes construct a Lyapunov set and verify Assumption 4 in higher-dimensional settings, it is obvious that such a direct construction and verification can become highly non-trivial when one considers multi-dimensional data that are not as simple. However, as mentioned in Remark 2.6, it is expected that the convex duality techniques of [11], which were developed in the context of the pure SP for the construction of Lyapunov sets that satisfy Assumption 4, would be useful in this context as well. Indeed, the result in Example 2.14(b) can be generalised using these techniques, but a detailed proof lies beyond the scope of this paper. The above results and examples only serve to demonstrate the importance of further developing and applying techniques that enable a systematic construction of these Lyapunov sets.

3 Stochastic dynamical systems

This section studies strong existence and pathwise uniqueness for a family of stochastic differential equations with reflection and a discontinuous drift. Let (G, D, F) be as in Sect. 2.2. In particular, F is defined by the right side of (1.10) with $f^c = 0$. We will apply the results of Sect. 2 to obtain existence and uniqueness for reflected SDE with drift $\tilde{F} = F + f^c$, where f^c is an arbitrary Lipschitz function. In preparation for the study, we establish in Sect. 3.1 the existence of nonanticipative selections for certain terms that appear in the CDMP. A standing assumption for this section will be the unique solvability of the CDMP for all RCLL trajectories, namely we assume:

Assumption 5 For every $T > 0$ and $u \in \mathcal{D}[0, T]$, there exists a unique $\phi \in \mathcal{M}(G, D, F, u)$.

As in Sect. 2, we use Φ to denote the solution map and if ϕ is in $\mathcal{M}(G, D, F, u)$, we write $\phi = \Phi(u)$. Existence and uniqueness questions for SDE will be taken up in Sect. 3.2 under the following strengthening of Assumption 5:

Assumption 6 For every $T > 0$ and $u \in \mathcal{D}[0, T]$, there exists a unique $\phi \in \mathcal{M}(G, D, F, u)$. Moreover, there exists $\kappa \in (0, \infty)$ such that

$$||\Phi(u_1) - \Phi(u_2)||_T \leq \kappa ||u_1 - u_2||_T,$$

for all $u_1, u_2 \in \mathcal{D}[0, T]$ and $T > 0$.

Recall that Theorem 2.11 gives sufficient conditions under which the above assumptions hold.

3.1 Nonanticipative selection

A map Ψ from $\mathcal{D}[0, T]$ to itself is said to be *nonanticipative* if for every $t \in [0, T]$,

$$u_1(s) = u_2(s) \quad \text{for } s \in [0, t] \Rightarrow \Psi(u_1)(s) = \Psi(u_2)(s) \quad \text{for } s \in [0, t].$$

Consider a CDMP (G, D, F) satisfying Assumption 5. An application of the uniqueness assumption to the time intervals $[0, t]$, $t \leq T$, directly implies that Φ is nonanticipative. Recall that a solution ϕ to the CDMP is of the form $\phi = u + \theta + \eta$, where $\theta = \int_0^{\cdot} \alpha_s \, ds$ and $\eta = \int_{[0,.]} \gamma_s \, d|\eta|_s$, and some additional conditions (as specified in Definition 1.1) are satisfied by α and γ. In this section we are interested in nonanticipative selections of the θ and η components of the solution. We begin with an example that demonstrates that these components may not be unique even when the solution ϕ is unique. Thus the existence of nonanticipative maps $u \mapsto \theta$ and $u \mapsto \eta$ does not follow immediately from the results of the previous section (unlike the case of the map $u \mapsto \phi$).

Example 3.1 Let $G = \mathbb{R}^2_+$ and consider the normal reflection field on ∂G. Let $C_1 = \{x \in G : x_1 < x_2\}$ and $C_2 = \{x \in G : x_1 > x_2\}$. Let $b_1 = -e_2$ and $b_2 = -e_1$. Assumption 3 is immediately satisfied since $n_i = d_i = e_i$ for $i = 1, 2$ and $b_1 - b_2 = \nu_{12} = e_1 - e_2$, where ν_{12} is the normal to the hyperplane $\partial C_1 \cap \partial C_2$ pointing away from C_1. It is also straightforward to check that $B = \{z \in \mathbb{R}^2 : |z_1| \leq 4, |z_2| \leq 4, |z_1 + z_2| \leq 6\}$ satisfies Assumption 4. In addition, since G is convex and the directions of constraint are normal, the usual normal projection π onto a convex set, satisfies Assumption 1. Lastly, Assumption 2 is also easily verified for this example (since the reflection matrix is clearly completely-$\mathcal{S}$ (see Remark 2.1)). Thus, by Theorem 2.11, there exists a unique solution $\phi = u + \theta + \eta$ for every $u \in D([0,T] : \mathbb{R}^2)$. In order to demonstrate multiple choices for (θ, η), consider $u = 0$. It follows immediately from the definition that $d(0) = S^1 \cap \mathrm{conv}(e_1, e_2) = S^1 \cap \mathbb{R}^2_+$ and $F(0) = \mathrm{conv}(-e_1, -e_2) = -\mathbb{R}^2_+$. It is clear that one can realise the solution $\phi = 0$ with $\theta(t) = -e_1 t$ and $\eta(t) = e_1 t$, and also with $\theta(t) = -e_2 t$ and $\eta(t) = e_2 t$ (or, in fact, with $\theta_t = vt$ for any $v \in d(0)$ and $\eta(t) = -vt$).

We first show that the singular part of η is unique, and identify a new quantity β, that will be useful in the next result.

Theorem 3.2 *Suppose that Assumption 5 holds for the polyhedral CDMP (G, D, F). Then for any $u \in \mathcal{D}[0,T]$ and $\phi = \Phi(u)$, there exists a unique $\zeta \in \mathcal{D}[0,T]$ and Lebesgue-a.e. unique $\beta \in \mathcal{L}^1[0,T]$ such that*

(i)
$$\phi(t) = u(t) + \int_0^t \beta(s)\, ds + \zeta(t), \quad t \in [0,T];$$

(ii)
$$\beta(t) \in F(\phi(t)) + D(\phi(t)), \ \textit{a.e. } t \in [0,T];$$

(iii) $|\zeta|(T) < \infty$ *and the measure* $d|\zeta|$ *is mutually singular with respect to Lebesgue measure on* $[0,T]$;

(iv) *there exists* $\tilde{\gamma} \in \mathcal{BM}[0,T]$ *such that* $\tilde{\gamma}(s) \in d(\phi(s))$ *for* $d|\zeta|$*-a.e.* $s \in [0,T]$ *and*
$$\zeta(t) = \int_{[0,t]} \tilde{\gamma}(s) 1\!\!1_{\{\phi(s) \in \partial G\}}\, d|\zeta|(s).$$

Moreover, $\tilde{\gamma}$ *in* (iv) *is* $|\zeta|$*-a.e. unique. Finally, the maps* $u \mapsto \zeta$ *and* $u \mapsto \int_0^\cdot \beta(s) ds$ *are nonanticipative.*

Proof Observe that the solution map Φ is well-defined on $\mathcal{D}[0,T]$ by Assumption 5. Fix $u \in \mathcal{D}[0,T]$ and $\phi = \Phi(u)$. We will first consider the uniqueness of ζ and β. Let $(\zeta, \beta) = (\zeta_1, \beta_1)$ and $(\zeta, \beta) = (\zeta_2, \beta_2)$ be two pairs satisfying properties (i)–(iv) of the theorem. From (i) above, we see that

$$\int_0^t \beta_1(s)\, ds + \zeta_1(t) = \int_0^t \beta_2(s)\, ds + \zeta_2(t), \quad t \in [0,T].$$

Due to (iii) above and the Radon–Nikodym theorem, this implies that $\zeta_1(t) = \zeta_2(t)$ and $\int_0^t \beta_1(s)\, ds = \int_0^t \beta_2(s)\, ds$ for all $t \in [0,T]$. This proves the asserted uniqueness of ζ

and β. Also, from (iv) we know that for all $t \in [0, T]$, $|\zeta|_t = \int_{[0,t]} 1\!\!1_{\{\phi(s)\in\partial G\}}\, d|\zeta|(s)$. This shows that $\tilde{\gamma}(s) = d\zeta(s)/d|\zeta|(s)$, $d|\zeta|$-a.e., and hence proves the asserted uniqueness of $\tilde{\gamma}$.

Next, we consider existence. Let θ, α, η and γ be as in Definition 1.1. By the Radon–Nikodym theorem, we can write

$$|\eta|(t) = \int_0^t \nu(s)\, ds + \lambda(t), \quad t \in [0, T] \tag{3.1}$$

for some $\nu \in \mathcal{L}^1[0, T]$ and for some non-decreasing function $\lambda : [0, T] \to \mathbb{R}_+$ which is singular with respect to Lebesgue measure on $[0, T]$. This shows that, for $t \in [0, T]$,

$$\begin{aligned}\eta(t) &= \int_{[0,t]} \gamma(s)\, d|\eta|(s) \\ &= \int_{[0,t]} \gamma(s)\nu(s)\, ds + \int_{[0,t]} \gamma(s)\, d\lambda(s).\end{aligned}$$

Define $\beta(t) \doteq \alpha(t) + \gamma(t)\nu(t)$ and $\zeta(t) \doteq \int_{[0,t]} \gamma(s)\, d\lambda(s)$. Since $\nu, \alpha \in \mathcal{L}^1[0, T]$ and $\gamma \in \mathcal{BM}[0, T]$, it follows that $\beta \in \mathcal{L}^1[0, T]$. Recalling that $\gamma(s) \in d(\phi(s))$, $d|\eta|$-a.e., from (3.1) we conclude that the same property holds $d\lambda$-a.e. as well. This, in particular, says that $|\zeta| = \lambda$. From Definition 1.1(iv), we know that $|\eta|(t) = \int_{[0,t]} 1\!\!1_{\{\phi(s)\in\partial G\}}\, d|\eta|(s)$. Along with (3.1), this shows that $\lambda(t) = \int_{[0,t]} 1\!\!1_{\{\phi(s)\in\partial G\}}\, d\lambda(s)$. It is now easy to check that with the above choice of β and ζ, (i) through (iv) hold. $\square$

The following result provides a nonanticipative selection of θ and η as a function of u.

Theorem 3.3 *Suppose that Assumption 5 holds for the polyhedral CDMP (G, D, F). Then there exist measurable, nonanticipative maps Θ and Π from $\mathcal{D}[0, T]$ to itself, such that for $u \in \mathcal{D}[0, T]$, $\Theta(u)$ and $\Pi(u)$ comprise the θ and, respectively, η components of the solution $\phi = \Phi(u)$.*

Proof A basic ingredient in the proof is the following statement:

$$\begin{aligned}&\text{for every } x \in G, \text{ there exist continuous maps} \\ &\ell_x^1 : F(x) + D(x) \to F(x), \quad \ell_x^2 : F(x) + D(x) \to D(x), \\ &\text{such that } \ell_x^1(\beta) + \ell_x^2(\beta) = \beta, \quad \forall \beta \in F(x) + D(x).\end{aligned} \tag{3.2}$$

The case $x \in G^\circ$ is trivial, since then $D(x) = \{0\}$, and one takes $\ell_x^1(\beta) = \beta$, $\ell_x^2(\beta) = 0$. Therefore, let $x \in \partial G$ be fixed and, in the present paragraph, suppress the symbol x from the notation. For $\beta \in F + D$, define

$$H(\beta) \doteq \left\{(m^1, m^2) \in F \times D : m^1 + m^2 = \beta\right\}.$$

Note that, for $\beta \in F + D$, $H(\beta)$ is a closed convex set, and let $\ell(\beta)$ denote the unique minimiser of $|m|$ over $m = (m^1, m^2) \in H(\beta)$. We first show that the map $\beta \mapsto |\ell(\beta)|$ is convex, and thus continuous, on $F + D$. Let $\beta_1, \beta_2 \in F + D$ and $\beta = \alpha\beta_1 + (1-\alpha)\beta_2$, some $\alpha \in [0, 1]$. Let $m = \alpha\ell(\beta_1) + (1-\alpha)\ell(\beta_2)$. Then $m \in H(\beta)$, and therefore $|\ell(\beta)| \leq |m| \leq \alpha|\ell(\beta_1)| + (1-\alpha)|\ell(\beta_2)|$, which establishes the claim. We now claim that the map $\beta \mapsto \ell(\beta)$ is also continuous. Indeed, given $\beta \in F + D$, let the sequence $\{\beta_n\} \subset F + D$ be such that $\beta_n \to \beta$, and set $m_n = \ell(\beta_n)$ and $m = \ell(\beta)$. The continuity of the map $\beta \mapsto |\ell(\beta)|$ shows that $|m_n| \to |m|$ and therefore, in particular, that the sequence $\{m_n\}$ is uniformly bounded. Therefore $\{m_n\}$ has a convergent subsequence, whose limit, which we denote by m_*, satisfies $|m_*| = |m|$. Moreover, since $F \times D$ is closed, $m_* \in H(\beta)$. The equality $m_* = m$ then follows from the uniqueness of the minimiser of $|m'|$ over $m' \in H(\beta)$, and thus $m_n \to m$ and the claim is established. The validity of (3.2) now follows by letting ℓ^1 and ℓ^2 be defined via $\ell = (\ell^1, \ell^2)$. Next, note that the dependence of $\ell_x(\beta)$ on x is only via $F(x)$ and $D(x)$, i.e., $\ell_x(\beta) = \ell_{x'}(\beta)$ if $(F(x), D(x)) = (F(x'), D(x'))$. Clearly, there are finitely many values that the sets $(F(x), D(x))$ take as x ranges over G, and for each $x_0 \in G$, $\{x \in G | (F(x), D(x)) = (F(x_0), G(x_0))\}$ is a measurable subset of G. As a result, using (3.2), $(x, \beta) \mapsto \ell_x(\beta)$ is a measurable map. Let $u \in \mathcal{D}[0, T]$ be given. Keeping the notation of Theorem 3.2, let $\phi = \Phi(u)$, and noting that $\beta(t) \in (F + D)(\phi(t))$ for a.e. t (by property (ii) of Theorem 3.2), let, for $i = 1, 2$, $\beta^i(t) = \ell^i_{\phi(t)}(\beta(t))$. Let $\alpha(t) = \beta^1(t)$ and $\theta(t) = \int_0^t \alpha(s)\, ds$. It follows from (3.2) that

$$\alpha(t) \in F(\phi(t)), \qquad \text{a.e. } t \in [0, T]. \tag{3.3}$$

It is also clear from the preceding discussion that α and θ are measurable. Next, let $\beta^\circ(t) = 0$ if $\beta^2(t) = 0$, and otherwise set $\beta^\circ(t) = |\beta^2(t)|^{-1}\beta^2(t)$. Then (3.2) implies that

$$\beta^\circ(t) \in d(\phi(t)), \qquad \text{a.e. on } \{t : \phi(t) \in \partial G\}. \tag{3.4}$$

Recall that by (ii) of Theorem 3.2,

$$\tilde{\gamma}(t) \in d(\phi(t)), \qquad d|\zeta|\text{−a.e. on } \{t : \phi(t) \in \partial G\}, \tag{3.5}$$

and set

$$\eta(t) = \int_0^t \beta^2(s)\, ds + \int_{[0,t]} \tilde{\gamma}(s)\, d|\zeta|(s).$$

By mutual singularity of $d|\zeta|$ and Lebesgue measure, there exists a Borel set $Q \subset [0, T]$ such that $\int_Q ds = \int_{Q^c} d|\zeta| = 0$. We also have

$$d|\eta|_s = |\beta^2(s)|\, ds + |\tilde{\gamma}(s)|\, d|\zeta|_s = |\beta^2(s)|\, ds + d|\zeta|_s. \tag{3.6}$$

Thus, with $\gamma = 1\!\!1_{Q^c}\beta^\circ + 1\!\!1_Q\tilde{\gamma}$ and using (3.4), (3.5) and the definition of β°, we have

$$\eta(t) = \int_{[0,t]} (1\!\!1_{Q^c}\beta^\circ(s) + 1\!\!1_Q\tilde{\gamma}(s))\, d|\eta|(s) = \int_{[0,t]} \gamma(s)\, d|\eta|(s), \tag{3.7}$$

$$\gamma(t) \in d(\phi(t)), \qquad d|\eta|\text{—a.e. on } \{t : \phi(t) \in \partial G\}. \tag{3.8}$$

Moreover, by Theorem 3.2(iv), $\int 1\!\!1_{\{\phi\in\partial G\}} d|\zeta| = |\zeta|$, and by (3.2), $\beta^2(t) \in D(\phi(t))$ a.e. $t \in [0, T]$. Combining this with the fact that $D(x) = \{0\}$ for $x \in G^\circ$ and relation (3.6), we obtain

$$\int_{[0,t]} 1\!\!1_{\{\phi(s)\in\partial G\}}\, d|\eta|(s) = \int_{[0,t]} 1\!\!1_{\{\phi(s)\in\partial G\}}(|\beta^2(s)|\, ds + d|\zeta|(s)) = |\eta|(t). \tag{3.9}$$

By (3.2) and Theorem 3.2(i), we conclude that

$$u + \theta + \eta = u + \int_0^{\cdot} (\beta^1(s) + \beta^2(s))\, ds + \zeta = u + \int_0^{\cdot} \beta(s)\, ds + \zeta = \phi. \tag{3.10}$$

Equations (3.3), (3.7), (3.8), (3.9) and (3.10) verify that the functions θ, α, η and γ defined above meet the requirements in the definition of a solution to the CDMP for u. Given u, we have defined θ and η in terms of the function β, which is unique only in an a.e. sense. However, since θ is given in an integral form as $\int \ell^1_{\phi(s)}(\beta(s))\, ds$, this suffices to determine θ uniquely. Similarly, since $\eta = \int \ell^2_{\phi(s)}(\beta(s))\, ds + \zeta$ and since ζ is uniquely determined, so is η. We have thus constructed measurable maps Π and Θ as stated. Finally, the nonanticipative property of these maps now follows from that of ϕ, ζ and $\int_0^{\cdot} \beta(s)ds$. □

3.2 Existence and uniqueness

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space supporting an n-dimensional Brownian motion W. Denote by $\{\mathcal{F}_t\}$ a right continuous $\mathbb{P}$-complete filtration such that $\{W_t\}$ is an $\mathcal{F}_t$-martingale. Denote by $\mathcal{PM}$ the class of $\{\mathcal{F}_t\}$-progressively measurable processes having RCLL sample paths $\mathbb{P}$-a.s. Recall the function f^c introduced in Sect. 1.3, assumed to be Lipschitz on G. We are given $\sigma : G \to \mathbb{R}^{n\times n}$, an initial condition $x \in G$, and a stochastic process $U \in \mathcal{PM}$, and are interested in a stochastic dynamical system that is, roughly speaking, a solution to the equation

$$X_t = x + \int_0^t f^c(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s + \theta_t + \eta_t + U_t,$$

where θ and η are the corresponding discontinuous media term and Skorokhod constraining term, respectively. More precisely, we seek a stochastic process X that satisfies:

$$\begin{aligned}&\textit{there exist } \mathbb{R}^n\textit{-valued processes } \alpha, \gamma \in \mathcal{PM}\\ &\textit{and } \theta \in \mathcal{AC}, \eta \in \mathcal{BV} \textit{ such that } \mathbb{P}\textit{-a.s.}\\ &\begin{cases} X_t \in G \quad \text{for all } t\\ X_t = x + \int_0^t f^c(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s + \theta_t + \eta_t + U_t,\\ \theta_t = \int_0^t \alpha_s\,ds, \quad \alpha_t \in F(X_t) \quad \text{a.e. } t,\\ \eta_t = \int_0^t \gamma_s\,d|\eta|_s, \quad |\eta|_t = \int_0^t 1\!\!1_{\{X_s\in\partial G\}}\,d|\eta|_s, \gamma_t \in d(X_t),\ d|\eta|\text{-a.e.}\end{cases}\end{aligned} \tag{3.11}$$

Clearly, under Assumption 5, a process satisfying (3.11) also satisfies $\mathbb{P}$-a.s.,

$$X = \Phi\left(x + \int_0^{\cdot} f^c(X_s)\,ds + \int_0^{\cdot} \sigma(X_s)\,dW_s + U\right). \tag{3.12}$$

Theorem 3.4 below proves strong existence and pathwise uniqueness of solutions to (3.11). One motivation for including the process U in the formulation above is because then, as a special case of Theorem 3.4, we obtain an extension of the uniqueness result of Theorem 2.11 (see Remark 3.6). Another motivation is to allow for the interpretation of U as being an external control.

Theorem 3.4 *Let Assumption 6 hold and assume that σ is Lipschitz on G. Then there exists a unique $\{\mathcal{F}_t\}$-adapted process X satisfying (3.11) on $[0, T]$ $\mathbb{P}$-a.s.*

Proof Let us begin by showing the existence of a unique $\{\mathcal{F}_t\}$-adapted process X satisfying (3.12). Consider first the case where, for some $M < \infty$, one has $|U(t)| \le M$ for all t, $\mathbb{P}$-a.s. Recall from Theorem 3.3 that Φ is nonanticipative. Therefore, if X is an $\{\mathcal{F}_t\}$-adapted process and the r.h.s. of (3.12) is denoted by $h(X)$, we have that $h(X)$ is also $\{\mathcal{F}_t\}$-adapted. Recall the notation $||X||_t = \sup_{s\in[0,t]} |X_s|$. The global Lipschitz property of Φ, f^c and σ imply that for some constant c depending only on Φ, f^c and σ,

$$E[||h(X)_s - h(X')_s||_t^2] \le c \int_0^t E[||X_r - X'_r||_s^2]\,ds, \quad t \in [0, T], \tag{3.13}$$

whenever X and X' are $\{\mathcal{F}_t\}$-adapted processes satisfying

$$E[||X||_T^2 + ||X'||_T^2] < \infty.$$

Given (3.13), it is standard to show the existence of a unique $\{\mathcal{F}_t\}$-adapted process X satisfying (3.12) (see, e.g., the Proofs of Theorems 5.2.5 and 5.2.9 of [17]). To remove the boundedness assumption on U, let $\tau_k = \inf\{s : |U(s)| \geq k\}$ and set $U^k(s) = U(s)1\!\!1_{[0,\tau_k)}(s)$, for $k \in \mathbb{N}$. Then for each k there is a unique process X^k for which (3.12) holds with U^k in place of U; moreover $\tau_k \uparrow \infty$ $\mathbb{P}$-a.s., and X^j, $j \geq k$, all agree on $[0, \tau_k)$, $k \in \mathbb{N}$. Thus the a.s. pointwise limit satisfies (3.12). Any two processes satisfying (3.12) agree on $[0, \tau_k)$, and thus follows the uniqueness statement.

To conclude, we must prove the existence of processes $\alpha, \gamma \in \mathcal{PM}$ that satisfy (3.11). Let $Y = \int_0^\cdot f^c(X_s)\,ds + \int_0^\cdot \sigma(X_s)\,dW_s + U_t$. By (3.12) and Theorem 3.3, $X = Y + \theta + \eta$, where $\theta = \Theta(Y)$ and $\eta = \Pi(Y)$. Using the definition of Φ, there exist maps $\bar{\alpha}, \bar{\gamma} : [0, T] \times \Omega \to \mathbb{R}^n$ and a full $\mathbb{P}$-measure set Ω_1 such that for all $\omega \in \Omega_1$ the following holds: the sample path of U is RCLL, $\bar{\alpha}$ and $\bar{\gamma}$ are Borel measurable maps from $[0, T] \to \mathbb{R}^n$, and

$$\theta = \int_0^\cdot \bar{\alpha}_s\,ds, \quad \bar{\alpha}_s \in F(X_s), \tag{3.14}$$

$$\eta = \int_{[0,\cdot]} \bar{\gamma}_s\,d|\eta|_s, \quad |\eta| = \int_{[0,\cdot]} 1\!\!1_{\{X_s \in \partial G\}}\,d|\eta|_s, \quad \bar{\gamma}_s \in d(X_s). \tag{3.15}$$

Since Π and Θ are measurable and nonanticipative, the versions $\theta 1\!\!1_{\Omega_1}$ and $\eta 1\!\!1_{\Omega_1}$ of θ and, respectively, η (still denoted as θ and η in the sequel) are measurable, adapted processes for which all sample paths are continuous, and, respectively, RCLL. As a result, θ and η lie in $\mathcal{PM}$. Let

$$\alpha_t = \liminf_{s \downarrow t} \frac{\theta_s - \theta_t}{s - t} \qquad t \in [0, T).$$

Then α_t is progressively measurable on $\{\mathcal{F}_{t+\varepsilon}\}$ for every $\varepsilon > 0$, and since $\{\mathcal{F}_t\}$ is continuous, it lies in fact in $\mathcal{PM}$. We say $t \in [0, T)$ is a point of increase for $|\eta|$ if $|\eta|(t, s) > 0$ for all $s \in (t, T)$. Then for $t \in [0, T)$, define

$$\gamma_t = \begin{cases} \liminf_{s \downarrow t} \dfrac{\eta_s - \eta_t}{|\eta|(s, t)} & \text{if } t \text{ is a point of increase for } |\eta|, \\ 0 & \text{otherwise} \end{cases}$$

and note that, in analogy with α, γ lies in $\mathcal{PM}$. Moreover, on $[0, T) \times \Omega_1$ we have $\int_0^\cdot \alpha_s\,ds = \int_0^\cdot \bar{\alpha}_s\,ds$ and $\int_0^\cdot \gamma_s\,d|\eta|_s = \int_0^\cdot \bar{\gamma}_s\,d|\eta|_s$. Hence (3.14) and (3.15) are satisfied by α (a.e.) and γ ($d|\eta|$-a.e.) a.s. It thus follows that $(\alpha, \gamma, \theta, \eta)$ satisfy (3.11) a.s.

□

Remark 3.5 Prior results on strong solutions to SDE with discontinuous coefficients include the following. Veretennikov [23] proves existence and uniqueness of strong solutions for multi-dimensional SDE under a uniform ellipticity assumption. The paper [24] proves strong existence and pathwise uniqueness in the case where the diffusion coefficient is uniformly elliptic with respect to a part of the variables, and where the drift coefficient satisfies certain continuity conditions (that do not hold for the cases studied in our paper). Existence and uniqueness of strong solutions was obtained in [5] for the particular case when the drift of the SDE arises from a maximal monotone map, without a nondegeneracy condition. We note that the current paper also allows for a degenerate diffusion coefficient, and in fact, the special case when the diffusion term is zero recovers the general CDMP.

In the case where G is a proper subset of $\mathbb{R}^n$ the only strong existence and uniqueness results that are available correspond to diffusions with continuous coefficients [2,9,10,19,20], with the exception of [5] which allows for the special case when the drift arises from a maximal monotone map and G is convex with a normal constraint vector field D.

Remark 3.6 In the case where U is deterministic and $\sigma = 0$, X is a solution to (3.11) if and only if $X \in \mathcal{M}(G, D, \tilde{F}, U)$, where $\tilde{F} = F + f^c$. Thus Theorem 3.4 above extends the uniqueness statement of Theorem 2.11 to cover nonzero f^c.

4 Appendix

Here, we present the proofs of Lemma 2.2 of Sect. 2.1, which was used to establish existence of solutions to the CDMP, and Lemma 2.8 of Sect. 2.2.1, which derives some elementary consequences of Assumption 4.

Proof of Lemma 2.2. Let $E \doteq \{x \in G : |x| \leq R\}$, where $R \geq \sup_n ||\phi_n||_T$, and let H be a compact subset of $\mathbb{R}^n$ containing $\cup_{x \in E} F(x)$. Denote $\alpha_n \doteq \dot{\theta}_n$. Then

$$\theta_n(t) = \int_{[0,t]} \alpha_n(s)\, ds, \quad \alpha_n(t) \in F(\phi_n(t)), \quad a.e.\ t \in [0, T]. \tag{4.1}$$

Let $S(t) \doteq E \times H \times [0, t]$ for $t \in (0, T]$, and denote $S = S(T)$. For $n \in \mathbb{N}$, define a probability measure m_n on S as follows: for $A \in \mathcal{B}(E \times H)$,

$$m_n(A \times [0, t]) \doteq \frac{1}{T} \int_{[0,t]} 1\!\!1_{\{(\phi_n(s), \alpha_n(s)) \in A\}}\, ds.$$

The compactness of S guarantees that the measures $\{m_n\}$ are tight. Hence along a subsequence, which we denote again by $\{m_n\}$, we must have $m_n \Rightarrow m$, where m is a probability measure on S. Observe that since $m_n(E \times H \times \cdot)$ is normalized Lebesgue measure on $[0, T]$, so is $m(E \times H \times \cdot)$. Thus

$$m\{(x, \alpha, s) \in S : \phi(s) \neq \phi(s-)\} = 0 \tag{4.2}$$

and, moreover, $m(\partial(S(t))) = m(E \times H \times \{t\}) = 0$ for every $t \in [0, T]$. The latter property shows that, as $n \to \infty$,

$$\int_{S(t)} \alpha\, m_n(dx\, d\alpha\, ds) \to \int_{S(t)} \alpha\, m(dx\, d\alpha\, ds).$$

Since $S(t) \subseteq S(T)$, this convergence is in fact uniform over $t \in [0, T]$. In turn, this shows that, as $n \to \infty$,

$$\theta_n(t) = \int_{[0,t]} \alpha_n(s)\, ds = T \int_{S(t)} \alpha m_n(dx\, d\alpha\, ds) \to T \int_{S(t)} \alpha\, m(dx\, d\alpha\, ds).$$

Since, by assumption, $\theta_n \to \theta$ as $n \to \infty$, this implies that

$$\theta(t) = T \int_{S(t)} \alpha\, m(dx\, d\alpha\, ds), \quad t \in [0, T]. \tag{4.3}$$

Now note that the set

$$\Sigma_1 \doteq \{(x, \alpha, s) \in S : \alpha \notin F(x)\}$$

is open and that $m_n(\Sigma_1) = 0$ for every $n \in \mathbb{N}$. Since $m_n \Rightarrow m$, we have that $m(\Sigma_1) = 0$. For $j \in \mathbb{N}$, define

$$\Sigma_2^j \doteq \{(x, \alpha, s) \in S : |x - \phi(s)| > 1/j \text{ or } |x - \phi(s-)| > 1/j\}$$

and let $\Sigma_2 \doteq \cup_{j\in\mathbb{N}} \Sigma_2^j$. Due to the J_1-convergence of ϕ_n to ϕ, we conclude that $m_n(\Sigma_2^j) = 0$ for all sufficiently large n. Since $\phi \in \mathcal{D}[0, T]$ implies Σ_2^j is open, we have $m(\Sigma_2^j) = 0$ for every $j \in \mathbb{N}$, and consequently, $m(\Sigma_2) = 0$. Together with (4.2) and the fact that $m(\Sigma_1) = 0$, this implies that

$$m\{(x, \alpha, s) \in S : x = \phi(s),\ \alpha \in F(\phi(s))\} = 1. \tag{4.4}$$

Decomposing m in terms of the normalised Lebesgue measure $m(E \times H \times \cdot)$ and the conditional stochastic kernels $m_s(dx, d\alpha)$, $s \in [0, T]$, we can use (4.4) to rewrite (4.3) as

$$\begin{aligned}
\theta(t) = T \int_{S(t)} \alpha\, m(dx\, d\alpha\, ds) &= \int_{[0,t]} \left(\int_{E\times H} \alpha\, m_s(dx, d\alpha) \right) ds \\
&= \int_{[0,t]} \left(\int_{F(\phi(s))} \alpha\, m_s(E, d\alpha) \right) ds \\
&= \int_{[0,t]} \alpha(s)\, ds,
\end{aligned}$$

where $\alpha \in \mathcal{BM}[0, T]$ is defined in the obvious manner. Finally, using the convexity of $F(\phi(s))$, we see that $\alpha(s) \in F(\phi(s))$ for a.e. $s \in [0, T]$, thus concluding the proof of the lemma. □

Proof of Lemma 2.8 The proof of (2.7) is exactly the same as that of Lemma 2.1 of [9]. Consider now (2.8). Let $(i, j) \in \mathcal{E}(\mathcal{C})$. By Assumption 3, we either have $b_{ij} = 0$, in which case (2.8) holds trivially, or we have that

$$\langle b_{ij}, \nu_{ij} \rangle > 0, \tag{4.5}$$

which we shall assume in what follows. Now, fix $z \in \partial B$ and $\vartheta \in \vartheta(z)$, and assume without loss of generality that $\langle z, \nu_{ij} \rangle \neq 0$. Define

$$y \doteq z - \frac{\langle z, \nu_{ij} \rangle}{\langle b_{ij}, \nu_{ij} \rangle} b_{ij}.$$

We claim that $y \in B$. Suppose the claim is false. Then, since 0 lies in the interior of B, there exists $\theta \in (0, 1)$ such that $\theta y \in \partial B$. Also, since $\langle \theta y, \nu_{ij} \rangle = 0$, we have from Assumption 4 that $\langle b_{ij}, \tilde{\vartheta} \rangle = 0$ for all $\tilde{\vartheta} \in \vartheta(\theta y)$. In addition, by the convexity of B, $\langle \tilde{\vartheta}, \theta y - x \rangle \leq 0$ for all $x \in B$. Combining the above observations, we see that $\langle \tilde{\vartheta}, \theta z - x \rangle = \langle \tilde{\vartheta}, \theta y - x \rangle \leq 0$ for all $x \in B$. Recalling that $\theta \in (0, 1)$, substituting $x = z$ in the last relation, we deduce that $\langle \tilde{\vartheta}, z \rangle \geq 0$, while substituting $x = 0$, we obtain $\langle \tilde{\vartheta}, z \rangle < 0$. This leads to a contradiction, thus proving that $y \in B$. Using the convexity of B once again, we obtain $\langle z - y, \vartheta \rangle \leq 0$. Lastly, using the definition of y and recalling that (4.5) holds, we obtain (2.8). □

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