# *Mirror Couplings and Neumann Eigenfunctions*

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ABSTRACT. We analyze a pair of reflected Brownian motions in a planar domain $D$, for which the increments of both processes form mirror images of each other when the processes are not on the boundary. We show that for $D$ in a class of smooth convex planar domains, the two processes remain ordered forever, according to a certain partial order. This is used to prove that the second eigenvalue is simple for the Laplacian with Neumann boundary conditions for the same class of domains.

## 1. INTRODUCTION

We will prove that the second eigenvalue for the Laplacian with Neumann boundary conditions is simple for a class of planar convex domains. We will also present some geometric properties of the corresponding eigenfunctions. The main tool that we use is a coupling of a pair of reflected Brownian motions in the domain, for which the increments of both processes form mirror images of each other when both processes are not on the boundary. This coupling, referred to as a *mirror* coupling, has been used before to study properties of Neumann Laplacian eigenfunctions (see [4], [7] and references therein) and, in particular, has been used in [3] to determine whether the second eigenvalue is simple. That paper was concerned with "lip domains" defined as follows. A lip domain is a bounded planar domain that lies between graphs of two Lipschitz functions with the Lipschitz constant 1. In particular, it has sharp "left" and "right" endpoints. The current work complements, in a sense, the results derived in [3], and shows that the technique based on mirror couplings is also applicable to a class of smooth planar domains. The earlier paper [4], that also used couplings in a similar context, showed that the second Neumann eigenvalue is simple in a convex planar domain if the domain is sufficiently long, namely, if the ratio of the diameter to width of the domain is greater than 3.06. If in addition we assume that the domain has a line of symmetry, the same conclusion can be reached if the ratio of the diameter to width of the domain is greater than 1.53 (see Proposition 2.4 of [4]). In the current paper



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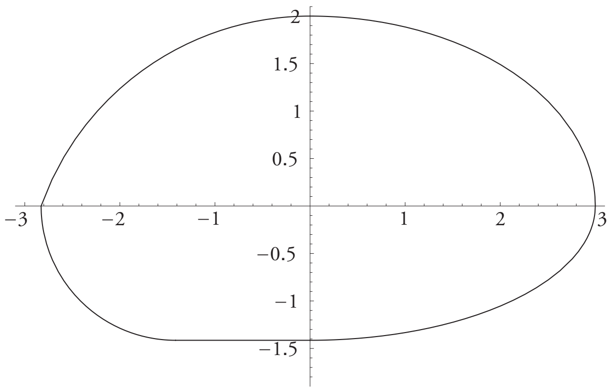



FIGURE 1.1. A domain with simple second Neumann eigenvalue.

we replace assumptions on the length to width ratio by a set of conditions that, in particular, allow us to obtain new results for domains that are not too long.

The motivation for this article comes from the “hot spots” conjecture of J. Rauch which states that the second Neumann eigenfunction attains its maximum on the boundary of the domain. The conjecture does not hold in full generality, see [5, 6, 9]. It does hold under a variety of extra assumptions, see [7] for a review of literature. This is related to the question of eigenvalue simplicity because it is often easier to analyze a single eigenfunction than a class of eigenfunctions. One technical approach to handle both the hot spots conjecture and the question of eigenfunction simplicity is first to change the problem to the mixed Neumann-Dirichlet problem by identifying the nodal line for the second eigenfunction (i.e., the line where the eigenfunction vanishes). This is easily done in symmetric domains (see [4, 10, 13]). Thus symmetry greatly simplifies the analysis of eigenfunctions, and removing symmetry from the assumptions is one of the main technical goals of this paper. The present paper is the first part of a project which aims at using this strategy for proving the hot spots conjecture for domains that are not necessarily symmetric.

The class of domains that we consider in this paper is defined via a number of geometric conditions. The conditions are elementary but their whole set is quite complicated so we will illustrate our main theorem with some examples. A domain that combines elements of “extreme” shapes compatible with our assumptions is depicted in Figure 1.1; see Example 5.1 for the analysis of this domain.

The set of conditions imposed on a domain $D$ is chosen so that for appropriately related reflected Brownian motions and an appropriate partial order, the two processes remain ordered in the same way forever. We call the line of symmetry for the two processes a “mirror.” We consider mirror couplings, i.e., pairs of reflected Brownian motions such that the increments of the two processes are symmetric images of each other with respect to the mirror, when both processes are in the interior of the domain. The mirror can be shown to perform a motion that is locally of bounded variation. The mirror does not move on any interval on which both processes remain in the interior of the domain. We analyze the motion of the mirror and construct an appropriate "Lyapunov set," i.e., a set with the property that the mirror remains in this set for all times, with probability one, provided that it starts inside the set. The partial ordering alluded to above is defined in terms of this set. An easy consequence of this property of the coupling is that there exists a second Neumann eigenfunction that is monotone with respect to the partial order. We do not know how to prove that the second eigenvalue is simple using standard results on positive linear operators such as the Krein-Rutman theorem—to do that, we would have to impose some extra assumptions on the domain $D$. We take an alternative approach, similar to that of [3]. Along with the partial order property alluded to above, this approach also uses crucially the following property of the coupling, which has a quite complex proof, see [3]. If the two processes are conditioned not to meet up to time 1, the conditional probability that their distance is greater than $c_1 > 0$ at time 1 is greater than $p_1 > 0$, where $c_1$ and $p_1$ do not depend on the starting points of the processes.

The paper is organized as follows. In the next section we list assumptions on the domains that we consider and state our main result. In Section 3, we review basic facts about reflected Brownian motions and mirror couplings. The same section contains the construction of the Lyapunov set and the proof that it is left invariant under the dynamics of the mirror process. Section 4 is devoted to the proof of our main result, Theorem 2.6. Section 5 presents some examples.

We are grateful to Rodrigo Bañuelos for very helpful advice.

## 2. Assumptions and the Main Result

In the first part of the paper, we consider a bounded strictly convex planar domain $D$ with $C^2$ boundary $\partial D$. We will later show that, in a suitable sense, one can remove the assumptions of strict convexity and $C^2$ smoothness (see the end of Section 4). For $A \in \partial D$ let $\mathbf{n}(A)$ denote the unit inward normal to $\partial D$ at $A$. For two distinct points $A$, $B$ in the plane, we denote by $[A, B]$ the closed line segment joining them, and by $\ell(A, B)$ the straight line containing them. We denote by $\mathcal{R}[A, B)$ the closed ray contained in $\ell(A, B)$, starting from $A$ and *not* containing $B$. We fix an orthonormal coordinate system with a basis $(\mathbf{e}_1, \mathbf{e}_2)$. We identify $\mathbb{R}^2$ and $\mathbb{C}$ and we use both types of notation for convenience. For any distinct points $A$ and $B$, $\angle(A, B)$ denotes the angle between $\mathbf{e}_1$ and $\ell(A, B)$, with the convention that $\angle(A, B) \in [0, \pi)$. We let $\mathbf{p}(A, B) = e^{i\angle(A,B)}$, and $\mathbf{m}(A, B) = -i\mathbf{p}(A, B)$. If $\ell$ is a line, we define $\angle\ell$, $\mathbf{p}(\ell)$ and $\mathbf{m}(\ell)$ by choosing any distinct points $A$, $B \in \ell$ and letting $\angle\ell = \angle(A, B)$, $\mathbf{p}(\ell) = \mathbf{p}(A, B)$ and $\mathbf{m}(\ell) = \mathbf{m}(A, B)$. Note that $\mathbf{m}(A, B) \cdot \mathbf{e}_1 = \mathbf{p}(A, B) \cdot \mathbf{e}_2 \geq 0$. For a point $A \in \partial D$, we let $\angle(A) \in [0, 2\pi)$ be defined by $\mathbf{n}(A) = e^{i\angle(A)}$.

The closed arc of $\partial D$ joining points $A$ and $B$ on the boundary is denoted by $\operatorname{arc}(A, B)$. When we use this notation, we will specify which one of the two arcs is meant unless it is clear from the context.

We now list our assumptions on the domain $D$. The assumptions that are most significant are labelled for future reference in the proofs.

We will use four sequences of points on the boundary: $P_1, P_2, \dots, P_6$, $Q_1, Q_2, \dots, Q_6$, $P_1', P_2', \dots, P_6'$, and $Q_1', Q_2', \dots, Q_6'$. In this section, we will only discuss points with subscripts 1, 3, 4 and 6. This is because we chose the notation so that each of these sequences is naturally ordered along the boundary, but the existence of points with subscripts 2 and 5 and some special properties will be proved only in Section 3.

We assume that there exists an angle $\alpha \in (0, \pi/2)$ such that all of the following conditions hold. Let $P_1 \in \partial D$ be such that $\mathbf{n}(P_1) = e^{i\alpha}$. Note that $P_1$ exists and is unique because $D$ is assumed to be strictly convex and $C^2$. Let $Q_1 \neq P_1$ be the unique point on the boundary for which $\angle(P_1, Q_1) = \alpha$ (see Figure 2.1(a)). Similarly, let $Q_6 \in \partial D$ denote the unique point with $\mathbf{n}(Q_1) = e^{-i\alpha}$ and $P_6 \in \partial D$ be such that $\angle(P_6, Q_6) = \alpha$. We assume that $(P_6 - P_1) \cdot \mathbf{e}_1 > 0$ and $(Q_6 - Q_1) \cdot \mathbf{e}_1 > 0$. We let $\alpha' = \pi - \alpha$ and define points $P_1'$, $Q_1'$, $P_6'$, and $Q_6'$ relative to $\alpha'$ in the same way that $P_1$, $Q_1$, $P_6$, and $Q_6$ have been defined relative to $\alpha$, and assume that $(P_6' - P_1') \cdot \mathbf{e}_1 < 0$.

Denote by $\partial_\uparrow D$ the closed arc of the boundary from $Q_6'$ to $Q_6$, not containing $P_1$. We refer to this arc as the *upper part of the boundary*. The arc $\operatorname{arc}(P_1, P_1')$ not containing $Q_6$ will be denoted $\partial_\downarrow D$ and referred to as the *lower part of the boundary*. For points $A, B \in \partial D$ we write $A < B$ if the first coordinate of $A$ is less than that of $B$. This ordering will only be used when both $A$ and $B$ are in $\partial_\uparrow D$ or when they are both in $\partial_\downarrow D$.

We say that a line $\ell$, or line segment $[A, B]$, is *admissible* if it intersects both $\partial_\uparrow D$ and $\partial_\downarrow D$, and $\angle \ell \in [\alpha, \alpha']$ (or $\angle(A, B) \in [\alpha, \alpha']$). For a line $\ell$ that is not horizontal, we say that a point $C \notin \ell$ is *on the left* of $\ell$ if there exist $D \in \ell$ and $a > 0$ such that $C + a\mathbf{e}_1 = D$. We say that a point is on the left of a line segment $[A, B]$ if it is on the left of $\ell(A, B)$. Points *on the right* are defined in an analogous way. Suppose $\ell$ is a line passing through $D$. We say that a boundary point $x \in \partial D \setminus \ell$ is *active for* $\ell$ if its reflection about $\ell$ is in $\bar{D}$. This seemingly strange term refers to mirror couplings defined in the next section.

We will state a number of assumptions for $P_1, P_2, \dots, P_6$ and $Q_1, Q_2, \dots, Q_6$. When we say that "an analogous condition holds for the primes" we mean that the analogous condition holds for $P_1', P_2', \dots, P_6'$ and $Q_1', Q_2', \dots, Q_6'$.

***Assumption 2.1.*** There exist line segments $[P_3, Q_3]$ and $[P_4, Q_4]$ satisfying $\angle(P_3, Q_3) = \angle(P_4, Q_4) = \alpha$ and such that $P_1 < P_3 < P_4 < P_6$. Moreover, if $[P, Q]$ is an admissible line segment with $P_1 < P < P_3$ and $\angle(P, Q) \geq \angle(P)$, then no right boundary point is active. If $[P, Q]$ is an admissible line segment with $Q_4 < Q < Q_6$ and $\angle(P, Q) \geq -\angle(Q)$, then no left boundary point is active. Analogous conditions hold for the primes.

Suppose that $\ell$ is a line that intersects $D$ and $A \in \partial D \setminus \ell$ is an active point. Let $\mathcal{T}$ denote the line tangential to $\partial D$ at $A$. If an intersection point of $\ell$ and $\mathcal{T}$ exists, it is said to be the *hinge* of $A$ at $\ell$ and it is denoted $H(A, \ell)$. If $\ell = \ell(P, Q)$, then $H(A, \ell)$ will be called the hinge of $A$ at $[P, Q]$. The name comes from the fact that the mirror $\ell$ for the coupling of reflected Brownian motions moves around $H(A, \ell)$ if one of these processes reflects at $A$ (see Section 3). We say that "hinge $H(A, \ell)$ does not exist" if $A$ is not an active point or $\ell$ and $\mathcal{T}$ are parallel.

If $P \in \partial_\downarrow D$, $Q \in \partial_\uparrow D$ and $H(A, \ell(P, Q)) \in \mathcal{R}[Q, P)$, then we say that the hinge is *upper*. Otherwise we say that it is *lower*. We say that $H(A, \ell(P, Q))$ is an *upper right hinge* if $A$ is on the right of $[P, Q]$ and $H(A, \ell(P, Q))$ is an upper hinge. We define upper left, lower right and lower left hinges in an analogous way.

***Assumption 2.2.*** There is $\nu > 0$ such that for all $P \in \partial_\downarrow D$ and $Q \in \partial_\uparrow D$ with $\angle(P, Q) \in [\alpha - \nu, \alpha]$ and $P_3 < P < P_4$, there exists no lower left and no upper right hinge. An analogous condition is assumed for the primes.

It follows from Assumption 2.3 below that $\text{arc}(P_3, P_4)$ is, in fact, the largest arc with the above property.

Since $D$ is strictly convex, $\alpha < \angle(P)$ for $P_1 < P < P_3$. We define $\mathcal{A}(P_1, P_3)$ as the set of line segments $[P, Q]$ with the properties $P_1 < P < P_3$ and $\angle(P, Q) \in (\alpha, \angle(P))$. We define analogously $\mathcal{A}(Q_4, Q_6)$, $\mathcal{A}(P'_1, P'_3)$ and $\mathcal{A}(Q'_4, Q'_6)$.

It is easy to see that for any $[P, Q] \in \mathcal{A}(P_1, P_3)$ there exists at least one lower right hinge. In fact, every $A \in \partial_\downarrow D$, $A > P$, that is sufficiently close to $P$ is active and the corresponding hinge is lower right.

***Assumption 2.3.*** For any line segment in $\mathcal{A}(P_1, P_3)$ there exists at least one lower left but no upper right hinge. For any line segment in $\mathcal{A}(Q_4, Q_6)$ there exists at least one upper right but no lower left hinge. Analogous conditions hold for the primes.

For an admissible $[P, Q]$, denote the right [resp., left] part of the boundary, excluding the endpoints $P$ and $Q$, by $\partial_R(P, Q)$ [$\partial_L(P, Q)$], and its reflection about $\ell(P, Q)$ by $\tilde{\partial}_R(P, Q)$ [$\tilde{\partial}_L(P, Q)$].

***Assumption 2.4.*** For every $[P, Q] \in \mathcal{A}(P_1, P_3)$ [resp., $\mathcal{A}(Q_4, Q_6)$], the curves $\partial_R(P, Q)$ and $\tilde{\partial}_L(P, Q)$ intersect at a unique point, and the intersection is nontangential. Moreover, both tangent lines to these curves at the point of intersection intersect $\mathcal{R}[P, Q)$ [resp., $\mathcal{R}[Q, P)$]. Analogous conditions hold for the primes.

Figure 2.2 illustrates a nontangential intersection of the boundary $\partial D$ and its reflection.

***Assumption 2.5.*** If $[P, Q] \in \mathcal{A}(P_1, P_3)$ and $[P', Q'] \in \mathcal{A}(Q'_4 Q'_6)$, then $\ell(P, Q) \cap \ell(P', Q')$ is non-empty and belongs to $\bar{D}$. An analogous statement holds for the pair $\mathcal{A}(P'_1, P'_3)$ and $\mathcal{A}(Q_4, Q_6)$.

Our main result is as follows.

***Theorem 2.6.*** *Assume that the set D satisfies all the conditions listed in this section, in particular, Assumptions 2.1–2.5. Then the second eigenvalue for the Laplacian in D with Neumann boundary conditions is simple.*

## 3. MIRROR COUPLING ANALYSIS

We start by a review of definitions and results from [3] on mirror couplings of reflected Brownian motions.

Let $W$ denote a standard planar Brownian motion and suppose that $x \in \bar{D}$. The equation

$$X(t) = x + W(t) + \int_0^t \mathbf{n}(X(s))\,\mathrm{d}\hat{L}(s),$$

where $\hat{L}$ denotes the local time of $X$ on the boundary, has a unique strong solution, referred to as a *reflected Brownian motion*. The local time does not increase when $X$ is away from the boundary of $D$, i.e., $\int_0^\infty 1_{\{X_s \in D\}}\,\mathrm{d}\hat{L}_s = 0$, a.s. . By a *coupling* we mean a pair of processes defined on the same probability space. We define a *mirror* coupling of reflected Brownian motions, denoted by $X$ and $Y$ and starting from $x, y \in \bar{D}$, by means of the following set of equations:

$$X(t) = x + W(t) + L(t), \quad L(t) = \int_0^t \mathbf{n}(X(s))\,\mathrm{d}\hat{L}(s), \tag{3.1}$$

$$Y(t) = y + Z(t) + M(t), \quad M(t) = \int_0^t \mathbf{n}(Y(s))\,\mathrm{d}\widehat{M}(s), \tag{3.2}$$

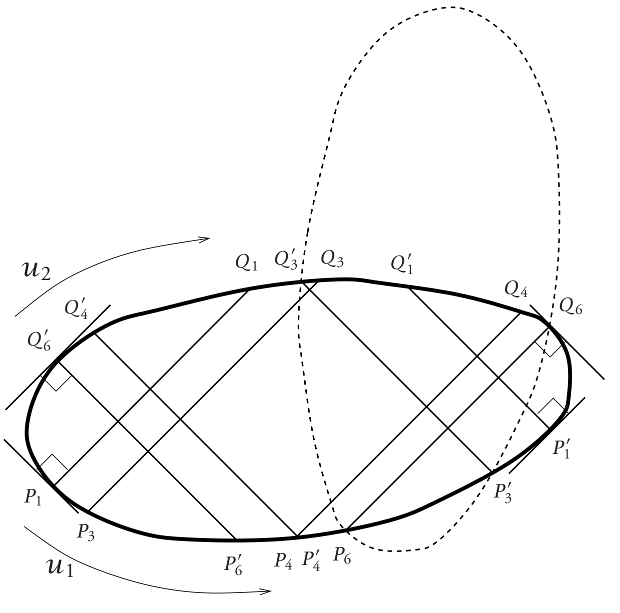
$$Z(t) = W(t) - 2\int_0^t \mathbf{m}(s)\mathbf{m}(s)\cdot \mathrm{d}W(s), \quad \mathbf{m}(t) = \frac{Y(t) - X(t)}{\|Y(t) - X(t)\|}. \tag{3.3}$$

Here $\widehat{M}$ stands for the local time of $Y$ on $\partial D$. The definition of $\mathbf{m}$ given above is different from the meaning given to this symbol in the previous section but the two vectors will be effectively identified in our arguments so no confusion should arise. The equations (3.1)-(3.3) have a unique strong solution up to the time $\zeta = \inf\{t : \lim_{s\to t-}(X(s) - Y(s)) = 0\}$ (see [3] for the precise meaning of this statement). The random variable $\zeta$ is called the *coupling time*. While $\{X(t) : t \geq 0\}$ is well defined by (3.1), the process $\mathbf{m}$, and consequently $Y$ is only well-defined on $[0, \zeta)$. We set

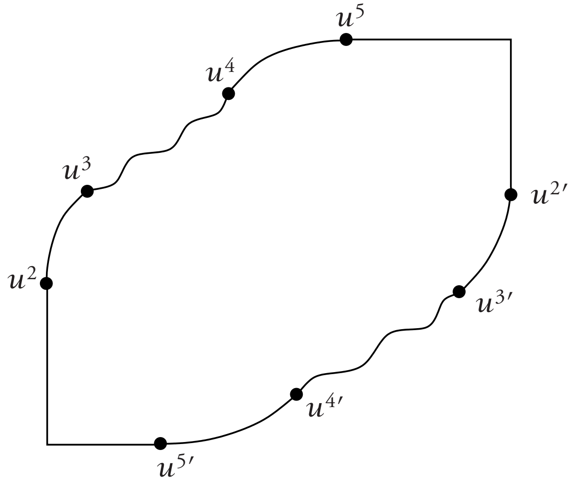
$$Y(t) = X(t) \quad \text{for all } t \geq \zeta. \tag{3.4}$$

Each of the processes $\{X(t) : t \geq 0\}$ and $\{Y(t) : t \geq 0\}$ is a reflected Brownian motion in $D$, and the pair $(X, Y)$ is a strong Markov process (cf. [3]).

So long as the processes $X$ and $Y$ have not coupled (i.e., for times $t < \zeta$), one can talk of a process $\ell(t)$, taking values in the set of lines in the plane and referred to as the *mirror* process, defined at time $t$ as the line with respect to which




(a) The domain $D$ is shown along with the special points on its boundary. A reflection of $D$ about $(P_3', Q_3')$ is shown in dashed line.




(b) A sketch of the set $\mathcal{L}$.

FIGURE 2.1.

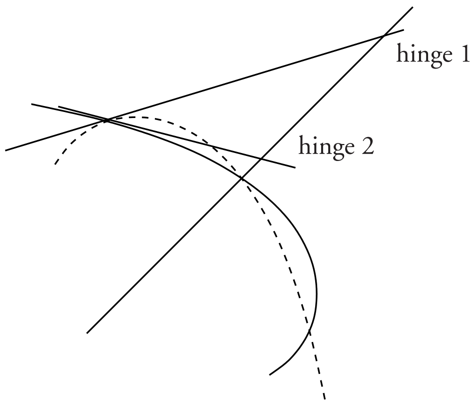



FIGURE 2.2. A mirror and two hinges.

$X(t)$ and $Y(t)$ are symmetric. Clearly, $\mathbf{m}(t)$ is a unit vector perpendicular to the mirror, by (3.3). It is also clear that, for each $t < \zeta$, $X(t)$ and $Y(t)$ cannot lie at any boundary point that is not active for $\ell(t)$. The main result of this section states that under the assumptions of Section 2 there is a nontrivial subset of $\bar{D} \times \bar{D}$ that is left invariant under the dynamics of the pair $(X, Y)$. It is more convenient to state and prove this result in terms of the motion of the mirror $\ell(t)$, a process that is locally of bounded variation.

We next develop an equation for the intersection points of the mirror with the boundary. Let $P(t)$ and $Q(t)$ denote the two intersection points of the mirror $\ell(t)$ with $\partial D$ (for $t < \zeta$). Let $\mathbf{p}(t) = \|Q(t) - P(t)\|^{-1}(Q(t) - P(t))$ and note that $\mathbf{p}(t)$ is orthogonal to $\mathbf{m}(t)$. We label the points in $\ell(t) \cap \partial D$ in such a way that $\mathbf{p}(t) = i\mathbf{m}(t)$. Recall how $\mathbf{p}(\ell)$ and $\mathbf{m}(\ell)$ have been defined in Section 2. If

$$(Q(t) - P(t)) \cdot \mathbf{e}_2 > 0, \tag{3.5}$$

both definitions of $\mathbf{p}$ and $\mathbf{m}$ are consistent in the sense that $\mathbf{p}(t) = \mathbf{p}(\ell(t))$ and $\mathbf{m}(t) = \mathbf{m}(\ell(t))$. This is the case, in particular, when $P(t) \in \partial_{\downarrow} D$ and $Q(t) \in \partial_{\uparrow} D$. Note that by convexity one has

$$\mathbf{p}(t) \cdot \mathbf{n}(P(t)) > 0, \quad \mathbf{p}(t) \cdot \mathbf{n}(Q(t)) < 0. \tag{3.6}$$

It will be convenient to work with the arclength parametrization of the boundary. If $A \in \partial_{\downarrow} D$, then we denote by $U_1(A)$ the length of the arc from $P_1$ to $A$ within $\partial_{\downarrow} D$. Analogously, if $A \in \partial_{\uparrow} D$, then we write $U_2(A)$ to denote the length of the arc from $Q_6'$ to $A$ within $\partial_{\uparrow} D$. We will write $U_1(t) = U_1(P(t))$ and $U_2(Q(t))$ if $P(t) \in \partial_{\downarrow} D$ and $Q(t) \in \partial_{\uparrow} D$. Let

$$\zeta_0 = \zeta \wedge \inf\{t \in [0, \zeta) : P(t) \notin \partial_{\downarrow} D \text{ or } Q(t) \notin \partial_{\uparrow} D\}.$$

The process $\{U(t) = (U_1(t), U_2(t)) : 0 \le t < \zeta_0\}$ uniquely identifies the mirror process $\ell(t)$ for $t \in [0, \zeta_0)$. Denote

$$V(t) = \|X(t) - Y(t)\|, \quad \theta(t) = \angle(\ell(t)), \quad \text{for } t < \zeta. \tag{3.7}$$

We will suppress the dependence on $t$ for all quantities in the following lemma.

***Lemma 3.1.*** *We have*

$$dU_1 = (\mathbf{p} \cdot \mathbf{n}(P)V)^{-1}(-(X-P) \cdot \mathrm{d}L + (Y-P) \cdot \mathrm{d}M), \tag{3.8}$$

$$dU_2 = (\mathbf{p} \cdot \mathbf{n}(Q)V)^{-1}((X-Q) \cdot \mathrm{d}L - (Y-Q) \cdot dM), \tag{3.9}$$

*and*

$$d\theta = V^{-1}\mathbf{p} \cdot (dM - dL), \tag{3.10}$$

*on the time interval* $[0, \zeta_0)$.

***Remark.*** Let

$$F = \Big( -(\mathbf{p}\cdot\mathbf{n}(P)V)^{-1}(X-P)\cdot\mathbf{n}(X),\ (\mathbf{p}\cdot\mathbf{n}(Q)V)^{-1}(X-Q)\cdot\mathbf{n}(X) \Big), \tag{3.11}$$

for $t \in [0, \zeta_0)$ for which $X(t) \in \partial D$ (in which case $\mathbf{n}(X)$ is well defined), and set $F = 0$ otherwise. Similarly, let

$$G = \Big( (\mathbf{p}\cdot\mathbf{n}(P)V)^{-1}(Y-P)\cdot\mathbf{n}(Y),\ -(\mathbf{p}\cdot\mathbf{n}(Q)V)^{-1}(Y-Q)\cdot\mathbf{n}(Y) \Big), \tag{3.12}$$

for $t$ such that $Y \in \partial D$ and $G = 0$ otherwise. We can write equations (3.8)–(3.9) in the form

$$dU = F\,\mathrm{d}|L| + G\,\mathrm{d}|M|. \tag{3.13}$$

*Proof.* By the results of [3], the process $\mathbf{m}$ satisfies

$$d\mathbf{m} = V^{-1}(dM - dL) - V^{-1}\mathbf{m}[\mathbf{m} \cdot (dM - dL)],$$

that can be written as

$$d\mathbf{m} = V^{-1}\mathbf{p}[\mathbf{p} \cdot (dM - dL)]. \tag{3.14}$$

Fix any $t_0 \ge 0$ and assume that $\{t_0 < \zeta_0\}$ holds. Let $\mathbf{n}_0 = \mathbf{n}(P(t_0))$ and $\mathbf{r}_0 = -i\mathbf{n}_0$. Let $J(t)$ be the intersection of $\ell(t)$ and the line tangential to $\partial D$ at $P(t_0)$. Set $x_1(t) = \mathbf{r}_0 \cdot (J(t) - P(t_0))$ and $m_1(t) = \mathbf{r}_0 \cdot \mathbf{m}(t)$. It follows from (3.14) that

$$dm_1 = V^{-1}[\mathbf{p} \cdot \mathbf{r}_0][\mathbf{p} \cdot (dM - dL)]. \tag{3.15}$$

Elementary geometry can be used to check that

$$x_1 = \frac{(X + Y - 2P(t_0)) \cdot \mathbf{m}}{2m_1}.$$

Applying Ito's formula to this representation of $x_1$ yields

$$dx_1 = (2m_1)^{-1}\mathbf{m} \cdot (dM + dL) + (2m_1)^{-1}(X + Y - 2P(t_0)) \cdot d\mathbf{m} - \frac{1}{2}m_1^{-2}[\mathbf{m} \cdot (X + Y - 2P(t_0))]\, dm_1. \tag{3.16}$$

Consider any vector $\mathbf{s}$. Since $\mathbf{p} \cdot \mathbf{r}_0 = -\mathbf{n}_0 \cdot \mathbf{m}$, we have

$$-\mathbf{s} \cdot \mathbf{m}(\mathbf{p} \cdot \mathbf{r}_0 + \mathbf{n}_0 \cdot \mathbf{m}) = 0.$$

We obtain in succession,

$$\begin{aligned}
&\mathbf{s} \cdot [-(\mathbf{p} \cdot \mathbf{r}_0)\mathbf{m} - (\mathbf{n}_0 \cdot \mathbf{m})\mathbf{m}] = 0,\\
&\mathbf{s} \cdot [m_1\mathbf{p} - (\mathbf{p} \cdot \mathbf{r}_0)\mathbf{m} - (\mathbf{n}_0 \cdot \mathbf{m})\mathbf{m} - (\mathbf{n}_0 \cdot \mathbf{p})\mathbf{p}] = 0,\\
&\mathbf{s} \cdot [m_1\mathbf{p} - (\mathbf{p} \cdot \mathbf{r}_0)\mathbf{m} - \mathbf{n}_0] = 0,\\
&m_1\mathbf{s} \cdot \mathbf{p} - [\mathbf{m} \cdot \mathbf{s}][\mathbf{p} \cdot \mathbf{r}_0] = \mathbf{n}_0 \cdot \mathbf{s}.
\end{aligned}$$

This, (3.14) and (3.15) imply that

$$m_1\mathbf{s} \cdot \mathrm{d}\mathbf{m} - \mathbf{m} \cdot \mathbf{s}\, \mathrm{d}m_1 = V^{-1}[\mathbf{n}_0 \cdot \mathbf{s}][\mathbf{p} \cdot (dM - dL)].$$

Next we substitute $\mathbf{s} = X + Y - 2P(t_0)$ to obtain

$$m_1[X + Y - 2P(t_0)] \cdot \mathrm{d}\mathbf{m} - \mathbf{m} \cdot [X + Y - 2P(t_0)]\, \mathrm{d}m_1 = V^{-1}[\mathbf{n}_0 \cdot (X + Y - 2P(t_0))][\mathbf{p} \cdot (dM - dL)],$$

and

$$(2m_1)^{-1}[X + Y - 2P(t_0)] \cdot \mathrm{d}\mathbf{m} - \frac{1}{2}m_1^{-2}\mathbf{m} \cdot [X + Y - 2P(t_0)]\, \mathrm{d}m_1 = (2m_1^2V)^{-1}[\mathbf{n}_0 \cdot (X + Y - 2P(t_0))][\mathbf{p} \cdot (dM - dL)].$$

We combine this with (3.16) to see that

$$dx_1 = (2m_1)^{-1}\mathbf{m} \cdot (dM + dL) + (2m_1^2V)^{-1}[\mathbf{n}_0 \cdot (X + Y - 2P(t_0))][\mathbf{p} \cdot (dM - dL)]. \tag{3.17}$$

Note that $m_1 = \mathbf{p} \cdot \mathbf{n}_0$ and at time $t = t_0$, the vector $\mathbf{p}$ is a positive multiple of $(Y + X)/2 - P$. Hence, for $t = t_0$,

$$m_1^{-1}[\mathbf{n}_0 \cdot ((Y + X)/2 - P)]\mathbf{p} - ((Y + X)/2 - P) = 0.$$

We obtain the following sequence of identities for $t = t_0$,

$$(Y - X)/2 + m_1^{-1}[\mathbf{n}_0 \cdot ((Y + X)/2 - P)]\mathbf{p} - (Y - P) = 0,$$

$$V\mathbf{m}/2 + m_1^{-1}[\mathbf{n}_0 \cdot ((Y + X)/2 - P)]\mathbf{p} - (Y - P) = 0,$$

$$(2m_1)^{-1}\mathbf{m} + (2m_1^2V)^{-1}[\mathbf{n}_0 \cdot (Y + X - 2P)]\mathbf{p} - (m_1V)^{-1}(Y - P) = 0,$$

$$(2m_1)^{-1}\mathbf{m} \cdot \mathrm{d}M + (2m_1^2V)^{-1}[\mathbf{n}_0 \cdot (Y + X - 2P)]\,\mathbf{p} \cdot \mathrm{d}M = (m_1V)^{-1}(Y - P) \cdot \mathrm{d}M,$$

$$(2m_1)^{-1}\mathbf{m} \cdot \mathrm{d}M + (2m_1^2V)^{-1}[\mathbf{n}_0 \cdot (Y + X - 2P)]\,\mathbf{p} \cdot \mathrm{d}M = ([\mathbf{p} \cdot \mathbf{n}_0]\,V)^{-1}(Y - P) \cdot \mathrm{d}M. \tag{3.18}$$

An analogous calculation yields

$$(2m_1)^{-1}\mathbf{m} \cdot \mathrm{d}L - (2m_1^2V)^{-1}[\mathbf{n}_0 \cdot (Y + X - 2P)]\mathbf{p} \cdot \mathrm{d}L = -([\mathbf{p} \cdot \mathbf{n}_0]V)^{-1}(X - P) \cdot \mathrm{d}L. \tag{3.19}$$

We combine (3.17)–(3.19) to obtain for $t = t_0$,

$$dx_1 = ([\mathbf{p} \cdot \mathbf{n}(P)]V)^{-1}[-(X - P) \cdot \mathrm{d}L + (Y - P) \cdot \mathrm{d}M].$$

The processes $x_1$ and $U_1$ satisfy $(dU_1/dx_1)(t_0) = 1$ because the boundary of $D$ is $C^2$. Therefore (3.8) follows. The proof of (3.9) is analogous.

Finally, from $\mathbf{p}(t) = i\mathbf{m}(t) = e^{i\theta(t)}$ it is easily seen that $d\theta = \mathbf{p} \cdot \mathrm{d}\mathbf{m}$, hence by (3.14) we obtain (3.10). □

***Construction of the Lyapunov set.*** We will construct a subset of the state space for mirrors (straight lines in the plane) with the property that if it contains $\ell(t)$, then it contains $\ell(s)$ for all $s \geq t$, a.s. . It is convenient to encode mirror positions using their intersection points with $\partial D$ and arclength parametrization $U_1$ and $U_2$, and so we will work with the process $U = (U_1, U_2)$ and a set $\mathcal{L} \subset \mathbb{R}^2$ in the state space of $U$. Going back to the assumptions and terminology of Section 2, if $\ell$ is an admissible line, let $P$ and $Q$ denote its intersection points with $\partial_\downarrow D$ and $\partial_\uparrow D$, and let $u_1 = U_1(P)$ and $u_2 = U_2(Q)$. Let $\bar{u}_1$ and $\bar{u}_2$ denote the length of $\partial_\downarrow D$ and $\partial_\uparrow D$. Then for $i = 1, 2$, $u_i$ takes values in $[0, \bar{u}_i]$. We will define the set

$\mathcal{L}$ as a subset of $\mathbf{U} = [0, \bar{u}_1] \times [0, \bar{u}_2]$. A point $u \in \mathcal{L}$ thus represents an admissible line $\ell$. The one-to-one (not onto) map from admissible line segments to points in $\mathbf{U}$ described above is denoted by $\varphi$, i.e., the image of $[P, Q]$ is $\varphi(P, Q)$. We use the notation $u^k = \varphi([P_k, Q_k])$ and $u^{k\prime} = \varphi([P'_k, Q'_k])$ for $k = 1, 2, \dots, 6$, where the special line segments with subscripts 1, 3, 4 and 6 were defined in Section 2, and those with subscripts 2 and 5 will be defined below.

The boundary of $\mathcal{L}$ consists of several pieces which will be described one by one. First, the following set will be a part of the boundary:

$$\operatorname{arc}(u^3, u^4) := \{\varphi(P, Q) : P_3 \le P \le P_4,\ \angle(P, Q) = \alpha\}.$$

Note that this subset of $\mathbf{U}$ is a curve connecting the points $u^3$ and $u^4$, corresponding to $[P_3, Q_3]$ and $[P_4, Q_4]$. See Figure 2.1(b).

Next we describe a curve that begins at the point $u^4$. To this end we will need the following lemma.

For $A \in \ell(P, Q) \setminus D$, let

$$d_{P,Q}(A) := \begin{cases} \|A - Q\|, & A \in \mathcal{R}[Q, P), \\ \|A - P\|, & A \in \mathcal{R}[P, Q). \end{cases}$$

***Lemma 3.2.*** *For every $[P, Q] \in \mathcal{A}(P_1, P_3)$, there exists a left boundary point $P_{\leftarrow} = P_{\leftarrow}(P, Q)$ with a lower hinge $H_{\leftarrow} = H_{\leftarrow}(P, Q)$ such that $d_{P,Q}(H_{\leftarrow}) \le d_{P,Q}(H)$ for every lower left hinge $H$, and there exists a right boundary point $P_{\rightarrow}$ with a lower hinge $H_{\rightarrow}$ such that $d_{P,Q}(H_{\rightarrow}) \ge (H)$ for every lower right hinge $H$. We also have*

$$d_{P,Q}(H_{\leftarrow}) > d_{P,Q}(H_{\rightarrow}). \tag{3.20}$$

*For $[P, Q] \in \mathcal{A}(Q_4, Q_6)$ there exist points $Q_{\leftarrow}$ and $Q_{\rightarrow}$ with properties analogous to $P_{\leftarrow}$ and $P_{\rightarrow}$.*

*Furthermore, for every $\varepsilon > 0$ there exists $C_\varepsilon < \infty$ with the following properties. Suppose that $[P, Q]$, $[\tilde{P}, \tilde{Q}] \in \mathcal{A}(P_1, P_3)$ are such that $\angle(P, Q) < \angle(P) - \varepsilon$, and assume that a similar inequality holds for $[\tilde{P}, \tilde{Q}]$. Then*

$$\|P_{\leftarrow} - \tilde{P}_{\leftarrow}\| + \|P_{\rightarrow} - \tilde{P}_{\rightarrow}\| \le C_\varepsilon(\|P - \tilde{P}\| + \|Q - \tilde{Q}\|), \tag{3.21}$$

*where $P_{\leftarrow} = P_{\leftarrow}(P, Q)$ and $\tilde{P}_{\leftarrow} = P_{\leftarrow}(\tilde{P}, \tilde{Q})$, etc. Similarly, if $[P, Q]$, $[\tilde{P}, \tilde{Q}] \in \mathcal{A}(Q_4, Q_6)$, $\angle(P, Q) < -\angle(Q) - \varepsilon$, and a similar inequality holds for $[\tilde{P}, \tilde{Q}]$, then*

$$\|Q_{\leftarrow} - \tilde{Q}_{\leftarrow}\| + \|Q_{\rightarrow} - \tilde{Q}_{\rightarrow}\| \le C_\varepsilon(\|P - \tilde{P}\| + \|Q - \tilde{Q}\|). \tag{3.22}$$

*Analogous results hold for the primes.*

*Proof.* Let $[P, Q] \in \mathcal{A}(P_1, P_3)$. Assumption 2.4 asserts the existence of a unique point of intersection of $\partial_R(P, Q)$ and $\tilde{\partial}_L(P, Q)$. Let $P_\rightarrow \in \partial_R(P, Q)$ denote this point, and let $P_\leftarrow(P, Q) \in \partial_L(P, Q)$ denote its reflection about $[P, Q]$. By Assumption 2.4, $P_\rightarrow$ has a lower right hinge, denoted by $H_\rightarrow$, and $P_\leftarrow$ has a lower left hinge, $H_\leftarrow$. Assumption 2.4 implies that all active points having lower right hinges must lie on the arc$(P, P_\rightarrow)$. Thus the inequality $d_{P,Q}(H_\rightarrow) \geq d_{P,Q}(H)$ for lower right hinges $H$ follows from convexity. Moreover, no active point having a lower left hinge can lie on the arc$(P_\leftarrow, P)$ (excluding $P_\leftarrow$), and thus by convexity, $d_{P,Q}(H_\leftarrow) \leq d_{P,Q}(H)$ for lower left hinges $H$. Since by Assumption 2.4 the intersection is nontangential, inequality (3.20) follows. Finally, let $\varepsilon > 0$ be given. For all line segments $[P, Q] \in \mathcal{A}(P_1, P_3)$ satisfying $\angle(P, Q) < \angle(P) - \varepsilon$, the intersection of $\partial_R(P, Q)$ and $\tilde{\partial}_L(P, Q)$ is nontangential, with a lower bound on the angle of intersection. Hence by smoothness of $\partial D$, the dependence of the point of intersection on $P$ and on $Q$ in this class is Lipschitz, with a constant depending only on $\varepsilon$. It follows from this and the definition of $P_\rightarrow$ and $P_\leftarrow$ that these two points are Lipschitz functions of $P$ and $Q$, with the Lipschitz constant depending only on $\varepsilon$. $\square$

For $u \in \mathbf{U}$ let $[P(u), Q(u)]$ denote the corresponding line segment with $P(u) \in \partial_\downarrow D$, and with an abuse of notation, let $\mathbf{p}(u) = \mathbf{p}(P(u), Q(u)) = e^{i\angle(P(u),Q(u))}$. Let $Q_\leftarrow(u)$ and $Q_\rightarrow(u)$ denote the boundary points defined relative to $[P(u), Q(u)]$ in Lemma 3.2 above. Note that $Q_\leftarrow(u)$ has an upper left hinge and $Q_\rightarrow(u)$ has an upper right hinge. We will prove in Lemma 3.3 below existence and some properties of a constant $a^* > 0$ and a curve

$$\{u(a) : a \in (0, a^*)\} \quad \text{in } \mathbf{U}$$

defined by the initial condition $u(0+) = u^4$ and the following set of ordinary differential equations,

$$\begin{aligned}(3.23)\quad \frac{d}{da}u_1 = \dot{u}_1 = (\mathbf{p}(u)\cdot\mathbf{n}(P(u)))^{-1}\Big[&-(Q_\leftarrow(u) - P(u))\cdot\mathbf{n}(Q_\leftarrow(u))\\ &-(Q_\rightarrow(u) - P(u))\cdot\mathbf{n}(Q_\rightarrow(u))\Big],\end{aligned}$$

$$\begin{aligned}(3.24)\quad \frac{d}{da}u_2 = \dot{u}_2 = (\mathbf{p}(u)\cdot\mathbf{n}(Q(u)))^{-1}\Big[&(Q_\leftarrow(u) - Q(u))\cdot\mathbf{n}(Q_\leftarrow(u))\\ &+(Q_\rightarrow(u) - Q(u))\cdot\mathbf{n}(Q_\rightarrow(u))\Big].\end{aligned}$$

These equations are obtained from (3.8)–(3.9) by formally replacing $U$ by $u$, $d|L|$ by $da$, $d|M|$ by $-da$, $V$ by 1, $X(t)$ by $Q_\leftarrow(u)$ and $Y(t)$ by $Q_\rightarrow(u)$. We note that we could have formally replaced $d|L|$ by $c_1\,da$ and $d|M|$ by $-c_2\,da$; that would not substantially alter the rest of the argument. The right hand sides of (3.23)–(3.24) are well defined for $u \in \varphi(\mathcal{A}(Q_4, Q_6))$ by Lemma 3.2. The number

$a^*$ has the property that the line $\varphi^{-1}(u(a))$ is asymptotically normal to $\partial D$ as $a \uparrow a^*$; see below for a precise statement. We denote by $u^5$ the limit $\lim_{a\uparrow a^*} u(a)$ (that exists by the result below) and denote $P_5$, $Q_5$ accordingly.

***Lemma 3.3.*** *There exists a unique constant $a^* \in (0,\infty)$ with the following properties. The equations (3.23)–(3.24) have a unique solution on $(0,a^*)$, and $u(a) \in \varphi(\mathcal{A}(Q_4,Q_6))$ on this interval. The limit $u^5 = \lim_{a\uparrow a^*} u(a)$ exists and one has $Q_4 < Q_5 < Q_6$, where $[P_5,Q_5] = \varphi^{-1}(u^5)$. Also, $\lim_{a\uparrow a^*} \mathbf{p}(u(a)) \cdot \mathbf{n}(Q(u(a))) = -1$, i.e., the line $\varphi^{-1}(u(a))$ is asymptotically normal to $\partial D$ at $Q_5$. Finally, the right hand sides of (3.23)–(3.24) are positive on $(0,a^*)$.*

*Proof.* By convexity of $D$, it follows that $\mathbf{p}\cdot\mathbf{n}(P) > 0$ and $\mathbf{p}\cdot\mathbf{n}(Q) < 0$ for $[P,Q] \in \mathcal{A}(Q_4,Q_6)$. For the same reason,

$$(3.25)\quad (Q_{\leftarrow}(u) - P(u))\cdot\mathbf{n}(Q_{\leftarrow}(u)) < 0, \quad (Q_{\rightarrow}(u) - P(u))\cdot\mathbf{n}(Q_{\rightarrow}(u)) < 0,$$

$$(Q_{\leftarrow}(u) - Q(u))\cdot\mathbf{n}(Q_{\leftarrow}(u)) < 0, \quad (Q_{\rightarrow}(u) - Q(u))\cdot\mathbf{n}(Q_{\rightarrow}(u)) < 0.$$

This shows that the right hand sides of (3.23) and (3.24) are strictly positive for $[P,Q] \in \mathcal{A}(Q_4,Q_6)$. Moreover, using the definition of $\mathcal{A}(Q_4,Q_6)$, one can see that the left hand side of the first inequality in (3.25) is bounded away from zero. As a result, the right hand side of (3.23) is bounded away from zero for $[P,Q] \in \mathcal{A}(Q_4,Q_6)$.

Let $\tilde{Q} \in \partial D$ be the point with $\mathbf{n}(\tilde{Q}) = ie^{i\alpha}$. By Assumptions 2.2 and 2.3 there are small perturbations of $[P_4,Q_4]$ for which there is no upper right hinge, and there are some for which there exists an upper right hinge. It is not hard to see that this implies that $Q_{\rightarrow}(u) \to \tilde{Q}$ as $u \to u^4$ along every sequence for which the hinge exists. We use this to extend the definition of $Q_{\rightarrow}(u)$, so that $Q_{\rightarrow}(u^4) = \tilde{Q}$. Consequently, the right hand sides of (3.23) and (3.24) are extended continuously to $\varphi(\mathcal{A}(Q_4,Q_6) \cup \{[P_4,Q_4]\})$. Let $\mathcal{A}_\varepsilon$ denote the set of line segments in $\mathcal{A}(Q_4,Q_6)$ having $\angle(P,Q) < -\angle(Q) - \varepsilon$. The local Lipschitz property asserted in (3.21)–(3.22) and the smoothness of $\mathbf{n}(\cdot)$ implies that the right hand sides of (3.23) and (3.24) are Lipschitz functions of $u$ for $u \in \varphi(\mathcal{A}_\varepsilon \cup \{[P_4,Q_4]\})$ (with a constant depending on $\varepsilon$). Let $a_\varepsilon = \inf\{a > 0 : u(a) \notin \varphi(\mathcal{A}_\varepsilon)\}$. The last assertion implies that for every $\varepsilon > 0$ there exists a unique solution on an interval $[0,a_\varepsilon)$, with the initial condition $u(0) = u^4$. Since by construction $\angle(P_4,Q_4) < -\angle(Q_4)$, and because the right hand sides are strictly positive, we have that $a_\varepsilon > 0$ for all small $\varepsilon > 0$. Since $u_1$ is bounded for $[P,Q] \in \mathcal{A}(Q_6,Q_6)$, it follows from the remark above regarding the right hand side of (3.23) being bounded away from zero, that $a_\varepsilon$ are bounded by a finite constant. The constants $a_\varepsilon$ are clearly monotone, the limit $a^* = \lim_{\varepsilon\to 0} a_\varepsilon$ exists and is finite. The solution to (3.23)–(3.24) on $[0,a^*)$ is thus well-defined and unique. We have already shown that the right hand sides of (3.23) and (3.24) are positive. Hence, $u_1$ and $u_2$ are monotone functions of $a$ and it follows that the limit $u^5 := \lim_{a\uparrow a^*} u(a)$ exists. We let $[P_5,Q_5] = \varphi^{-1}(u^5)$.

We will show that $Q_5 < Q_6$. It follows from (3.23)–(3.24) that

$$\frac{du_1}{du_2} = -\frac{\mathbf{p}\cdot\mathbf{n}(Q)}{\mathbf{p}\cdot\mathbf{n}(P)}\cdot\frac{(Q_\leftarrow - P)\cdot\mathbf{n}(Q_\leftarrow) + (Q_\rightarrow - P)\cdot\mathbf{n}(Q_\rightarrow)}{(Q_\leftarrow - Q)\cdot\mathbf{n}(Q_\leftarrow) + (Q_\rightarrow - Q)\cdot\mathbf{n}(Q_\rightarrow)}. \tag{3.26}$$

We can consider this as an equation for $u_1$ as a function of $u_2$ with the initial condition $u_1|_{u_2=u_2^4} = u_1^4$. For comparison, we consider a curve $v(a) = v = (v_1, v_2)$ in $\mathbf{U}$ with the property that $\angle(\varphi^{-1}(v)) = \alpha$ for all $a > 0$. This curve satisfies

$$\frac{dv_1}{dv_2} = -\frac{\mathbf{p}\cdot\mathbf{n}(Q)}{\mathbf{p}\cdot\mathbf{n}(P)}.$$

Again, we can consider the above as an equation for $v_1$ as a function of $v_2$, with the same initial condition as for $u_1$, namely, $v_1|_{v_2=u_2^4} = u_1^4$ (this is because $\angle(\varphi^{-1}(u^4)) = \alpha$). The fact that $Q_\rightarrow$ has an upper hinge implies that $(Q - P)\cdot\mathbf{n}(Q_\rightarrow) < 0$. Similarly, $(Q - P)\cdot\mathbf{n}(Q_\leftarrow) < 0$. It follows that the second fraction on the right hand side of (3.26) is strictly less than 1. Standard comparison results for univariate ODE's imply that $u_2 < v_2$ whenever $u_1 = v_1$. This shows that

$$\angle\Big(\varphi^{-1}(u(a))\Big) > \alpha = \angle(P_6, Q_6) \quad \text{for every } a \in (0, a^*]. \tag{3.27}$$

We are in the middle of an argument that is supposed to show that $Q_5 < Q_6$. We now argue by contradiction and assume that $Q_5 \ge Q_6$. Then $[P, Q_6] = \varphi^{-1}(u(\hat{a}))$ for some $\hat{a} \in (0, a^*]$ and $P$. By (3.27), $\angle(\varphi^{-1}(u(\hat{a})) > \alpha = -\angle(Q(u(\hat{a})))$. Hence for small $\varepsilon > 0$,

$$\angle\Big(\varphi^{-1}(u(\tilde{a}))\Big) > -\angle\Big(Q(u(\tilde{a}))\Big)$$

for an appropriate $\tilde{a} < a_\varepsilon$. This contradicts the definition of $a_\varepsilon$. We conclude that $Q_5 < Q_6$.

Finally, note that the limit $\lim_{a\uparrow a^*} \mathbf{p}(u(a))\cdot\mathbf{n}(Q(u(a)))$ exists by monotonicity of $u$ and is equal to $\mathbf{p}(u(a^*))\cdot\mathbf{n}(Q(u(a^*)))$. Since by (3.27) we have $\angle(P_5, Q_5) > \alpha$, and since $Q_4 < Q_5 < Q_6$, it follows from the definitions of $\mathcal{A}(Q_4, Q_6)$, $\mathcal{A}_\varepsilon$ and $a_\varepsilon$ that for all small $\varepsilon > 0$,

$$\angle\Big(\varphi^{-1}(u(a_\varepsilon))\Big) = -\angle\Big(Q(u(a_\varepsilon))\Big) - \varepsilon\,.$$

Thus $\angle(P_5, Q_5) = -\angle(Q_5)$ i.e., $\mathbf{p}(u(a^*))\cdot\mathbf{n}(Q(u(a^*))) = -1$. □

The part of the boundary constructed above is denoted by $\mathrm{arc}(u^4, u^5)$.

Analogously to $\mathrm{arc}(u^3, u^4)$, we construct $\mathrm{arc}(u^{3\prime}, u^{4\prime})$. Similarly to $\mathrm{arc}(u^4, u^5)$, we construct $\mathrm{arc}(u^{4\prime}, u^{5\prime})$, and then $\mathrm{arc}(u^2, u^3)$ and $\mathrm{arc}(u^{2\prime}, u^{3\prime})$.

Next, consider the two line segments $[P_5, Q_5]$ and $[P_2', Q_2']$. Since $Q_5 < Q_6$, Assumption 2.5 implies that these two line segments intersect in $D$. As a result, $P_5 < P_2'$ and for similar reasons, $Q_2' < Q_5$. We add the following pieces to the boundary of $\mathcal{L}$:

$$\{\varphi(P, Q_5) : P_5 \le P \le P_2'\}, \quad \{\varphi(P_2', Q) : Q_2' \le Q \le Q_5\}.$$

We denote this by $\operatorname{arc}(u^5, u^{2\prime})$. Note that it has the form

$$\{(u_1, u_2) \in \mathbf{U} : u_1^5 \le u_1 \le u_1^{2\prime},\ u_2 = u_2^5\} \\ \cup \{(u_1, u_2) \in \mathbf{U} : u_1 = u_1^{2\prime},\ u_2^{2\prime} \le u_2 \le u_2^5\}.$$

Finally, we construct $\operatorname{arc}(u^2, u^{5\prime})$, the last part of the boundary of $\mathcal{L}$, in a way analogous to the construction of $\operatorname{arc}(u^5, u^{2\prime})$.

In view of Lemma 3.3 it is easy to see that the pieces of $\partial\mathcal{L}$ constructed above do not intersect each other, except for the endpoints. The Lyapunov set $\mathcal{L}$ is defined as the simply connected, bounded, closed domain with the boundary comprised of all arcs constructed above.

***Invariance of the set*** $\mathcal{L}$**.** Recall the definitions of the mirror coupling $(X(t), Y(t))$, and $U(t)$, $\ell(t)$ and $\zeta$ from the beginning of this section. The main result of this section states that the process $U$ remains in $\mathcal{L}$ if it start in $\mathcal{L}$.

***Theorem 3.4.*** *Assume that $X(0) \ne Y(0)$, $U(0) \in \mathcal{L}$ and that $X(0)$ is on the left of $\ell(0)$. Then, with probability 1, for all $t < \zeta$,*

$$U(t) \in \mathcal{L} \quad \text{and} \quad \mathbf{e}_1 \cdot (Y(t) - X(t)) > 0.$$

*Proof.* Suppose for the moment that $U(t) \in \mathcal{L}$ for all $t < \zeta$. Then the assertion that $\mathbf{e}_1 \cdot (Y(t) - X(t)) > 0$ for $t < \zeta$ follows from sample path continuity of $X$ and $Y$ and the fact that $\ell(t) \in [\alpha, \pi - \alpha]$, $t < \zeta$. Hence, it remains to show that $U(t) \in \mathcal{L}$ for all $t < \zeta$.

We define $\tau = \inf\{t \in [0, \zeta) : U(t) \notin \mathcal{L}\}$ and $E = \{\tau < \zeta\}$, with the convention $\inf \varnothing = +\infty$. We will show that $\mathbb{P}(E) = 0$. On $E$, we let $u^* = U(\tau) \in \partial\mathcal{L}$. Note that $U(t_k) \in \mathcal{L}^c$ along a sequence $t_k \to \tau+$.

Consider first the case when $u^* \in \operatorname{arc}(u^4, u^5) \setminus \{u^4, u^5\}$. By (3.23)–(3.24), the vector $\mathbf{t}^* = (\mathbf{t}_1^*, \mathbf{t}_2^*)$ given by the following formula is tangent to $\partial\mathcal{L}$ at $u^*$,

$$\mathbf{t}_1^* = (\mathbf{p}^* \cdot \mathbf{n}(P^*))^{-1}[-(Q_{\leftarrow}^* - P^*) \cdot \mathbf{n}(Q_{\leftarrow}^*) - (Q_{\rightarrow}^* - P^*) \cdot \mathbf{n}(Q_{\rightarrow}^*)], \tag{3.28}$$

$$\mathbf{t}_2^* = (\mathbf{p}^* \cdot \mathbf{n}(Q^*))^{-1}[(Q_{\leftarrow}^* - Q^*) \cdot \mathbf{n}(Q_{\leftarrow}^*) + (Q_{\rightarrow}^* - Q^*) \cdot \mathbf{n}(Q_{\rightarrow}^*)]. \tag{3.29}$$

The superscripts * in the above formula indicate that all the functions are evaluated at $u^*$. Note that $V^* > 0$ since otherwise we would have $\tau = \zeta$. By Lemma

3.3, $\mathbf{t}_i^* > 0$, $i = 1, 2$. Thus $\mathbf{N}^* := (\mathbf{t}_2^*, -\mathbf{t}_1^*)$ is an inward normal vector to $\partial\mathcal{L}$ at $u^*$. It is not necessarily true that $\|\mathbf{N}^*\| = 1$. Note that $X^* \in \partial D$ or $Y^* \in \partial D$ (or both) because $X$ and $Y$ are continuous and the mirror $\ell(t)$ is not moving when the reflected Brownian motions are inside $D$. In the case when $X^* \in \partial D$, the expression on the right hand side of (3.11) evaluated at $\tau$ will be denoted by $F^*$. Similarly, in the case $Y^* \in \partial D$, $G^* := G(\tau)$ (cf. (3.12)).

We will now show that $F^* \cdot \mathbf{N}^* > 0$ in the case $X^* \in \partial D$. Let

$$\gamma^* = -\Big([\mathbf{p}^* \cdot \mathbf{n}(P^*)][\mathbf{p}^* \cdot \mathbf{n}(Q^*)]V^*\Big)^{-1},$$

and note that $\gamma^* > 0$. Since $\mathbf{N}^* = (\mathbf{t}_2^*, -\mathbf{t}_1^*)$,

$$\begin{aligned} (3.30)\quad & F_1^*\mathbf{t}_2^* - F_2^*\mathbf{t}_1^* \\ &= \gamma^*\Big\{(X^*-P^*)\cdot\mathbf{n}(X^*)[(Q_{\leftarrow}^*-Q^*)\cdot\mathbf{n}(Q_{\leftarrow}^*) + (Q_{\rightarrow}^*-Q^*)\cdot\mathbf{n}(Q_{\rightarrow}^*)] \\ &\quad - (X^*-Q^*)\cdot\mathbf{n}(X^*)[(Q_{\leftarrow}^*-P^*)\cdot\mathbf{n}(Q_{\leftarrow}^*) + (Q_{\rightarrow}^*-P^*)\cdot\mathbf{n}(Q_{\rightarrow}^*)]\Big\}. \end{aligned}$$

By Assumption 2.3 the hinges corresponding to $X^*$ and to $Q_{\leftarrow}^*$ are upper; thus $\mathbf{p}^* \cdot \mathbf{n}(X^*) < 0$ and $\mathbf{p}^* \cdot \mathbf{n}(Q_{\leftarrow}^*) < 0$. Also, Lemma 3.2 states that the distance from the hinge corresponding to $Q_{\leftarrow}^*$ to $Q^*$ is not smaller than the distance from the hinge corresponding to $X^*$ to $Q^*$. It follows that the distance from the hinge corresponding to $Q_{\leftarrow}^*$ to $P^*$ is not smaller than the distance from the hinge corresponding to $X^*$ to $P^*$. One can express this fact by the following inequality:

$$\frac{(Q_{\leftarrow}^* - P^*)\cdot\mathbf{n}(Q_{\leftarrow}^*)}{\mathbf{p}^*\cdot\mathbf{n}(Q_{\leftarrow}^*)} \geq \frac{(X^* - P^*)\cdot\mathbf{n}(X^*)}{\mathbf{p}^*\cdot\mathbf{n}(X^*)}.$$

Since $Q^* - P^*$ is a positive multiple of $\mathbf{p}^*$, it follows that

$$\begin{aligned} &[(Q_{\leftarrow}^*-P^*)\cdot\mathbf{n}(Q_{\leftarrow}^*)][(Q^*-P^*)\cdot\mathbf{n}(X^*)] \\ &\qquad \geq [(X^*-P^*)\cdot\mathbf{n}(X^*)][(Q^*-P^*)\cdot\mathbf{n}(Q_{\leftarrow}^*)], \\ &[(Q_{\leftarrow}^*-P^*)\cdot\mathbf{n}(Q_{\leftarrow}^*)][(Q^*-P^*)\cdot\mathbf{n}(X^*) - (X^*-P^*)\cdot\mathbf{n}(X^*)] \\ &\qquad \geq [(X^*-P^*)\cdot\mathbf{n}(X^*)][(Q^*-P^*)\cdot\mathbf{n}(Q_{\leftarrow}^*) - (Q_{\leftarrow}^*Z-P^*)\cdot\mathbf{n}(Q_{\leftarrow}^*)]. \end{aligned}$$

This gives

$$\begin{aligned} (3.31)\quad & [(X^*-P^*)\cdot\mathbf{n}(X^*)][(Q_{\leftarrow}^*-Q^*)\cdot\mathbf{n}(Q_{\leftarrow}^*)] \\ &\qquad - [(X^*-Q^*)\cdot\mathbf{n}(X^*)][(Q_{\leftarrow}^*-P^*)\cdot\mathbf{n}(Q_{\leftarrow}^*)] \geq 0. \end{aligned}$$

Next, by Assumption 2.3 and Lemma 3.2 (applied to $\mathcal{A}(Q_4, Q_6)$), $Q_{\rightarrow}^*$ has an upper right hinge. By Lemma 3.2, the distance of this hinge from $Q^*$ is strictly larger than that of the hinge corresponding to $Q^*_{\leftarrow}$, and, in turn, that corresponding to $X^*$. It follows that the distance of the hinge corresponding to $Q^*_{\rightarrow}$ from $P^*$ is strictly larger than that of the hinge corresponding to $X^*$. This can be written as

$$\frac{(Q^*_{\rightarrow} - P^*) \cdot \mathbf{n}(Q^*_{\rightarrow})}{\mathbf{p}^* \cdot \mathbf{n}(Q^*_{\rightarrow})} > \frac{(X^* - P^*) \cdot \mathbf{n}(X^*)}{\mathbf{p}^* \cdot \mathbf{n}(X^*)}. \tag{3.32}$$

A calculation similar to the one leading to (3.31) yields the strict inequality

$$\begin{aligned}[(X^* - P^*) \cdot \mathbf{n}(X^*)][(Q^*_{\rightarrow} - Q^*) \cdot \mathbf{n}(Q^*_{\rightarrow})] \\ - [(X^* - Q^*) \cdot \mathbf{n}(X^*)][(Q^*_{\rightarrow} - P^*) \cdot \mathbf{n}(Q^*_{\rightarrow})] > 0.\end{aligned} \tag{3.33}$$

We add (3.31) and (3.33) and combine the result with (3.30) to obtain $F^* \cdot \mathbf{N}^* > 0$.

A similar calculation (that is slightly more complicated due to the fact that $Y^*$ can either have an upper or a lower hinge) results in the conclusion that $G^* \cdot \mathbf{N}^* > 0$ in the case $Y^* \in \partial D$.

We now go back to (3.11)–(3.13). By the sample path continuity of the processes $|L|$, $|M|$, $X$, $Y$, $\mathbf{p}$, $P$, $Q$, and the continuity of the vector field $\mathbf{n}$ on $\partial D$, the fact that $F^* \cdot \mathbf{N}^* > 0$ provided that $X^* \in \partial D$ implies that there exists a (random) $\varepsilon > 0$ such that for all $\delta > 0$ small enough,

$$\int_{[\tau,\, \tau+\delta]} [F(t) \cdot \mathbf{N}^*]\, \mathrm{d}|L|(t) > \varepsilon |L|([\tau,\, \tau+\delta]).$$

Similarly, for sufficiently small $\delta > 0$,

$$\int_{[\tau,\, \tau+\delta]} [G(t) \cdot \mathbf{N}^*]\, \mathrm{d}|M|(t) > \varepsilon |M|([\tau,\, \tau+\delta]). \tag{3.34}$$

Thus by (3.13), for all sufficiently small $\delta > 0$,

$$(U(\tau + \delta) - U(\tau)) \cdot \mathbf{N}^* \geq \varepsilon \lambda(\delta), \tag{3.35}$$

where $\lambda(s) = |L|([\tau,\, \tau+s]) + |M|([\tau,\, \tau+s])$.

The boundary of $\mathcal{L}$ is $C^1$ in a neighborhood of $u^*$ so for any sequence $\hat{u}_k \in \mathcal{L}^c$ with $\hat{u}_k \to u^*$, we have $\limsup_{k\to\infty} (\hat{u}_k - u^*) \cdot \mathbf{N}^* / \|\hat{u}_k - u^*\| \leq 0$. Since $U(t_k) \in \mathcal{L}^c$ for a sequence $t_k \to \tau+$, we have $\lambda(s) > 0$ for $s > 0$. The last two observations imply that

$$(U(t_k) - U(\tau)) \cdot \mathbf{N}^* \leq C\lambda(t_k) r(t_k),$$

for some (random) $C < \infty$ and $r(t)$ such that $r(0+) = 0$. This contradicts (3.35); thus the probability that $U$ exits $\mathcal{L}$ through $u^* \in \text{arc}(u^4, u^5) \setminus \{u^4, u^5\}$ is equal to zero.

Next consider the case when $U$ exits $\mathcal{L}$ through $\text{arc}(u^5, u^{2\prime})$, excluding the points $u^5$ and $u^{2\prime}$. By Assumption 2.1, $X(\tau)$ is not on the boundary. Thus

$$(3.36)\qquad |L|([\tau-\varepsilon,\ \tau+\varepsilon]) = 0 \quad \text{for an appropriate random } \varepsilon > 0.$$

Hence only the term $G\,\mathrm{d}|M|$ is present in (3.13). It is easy to see from (3.12) that $G_i^* < 0$ for both $i = 1, 2$. The inward unit normals to $\partial\mathcal{L}$ at $u^*$ can be either $(0, -1)$ or $(-1, 0)$, except there is a single point (corner) where any convex combination of these vectors points inside $\mathcal{L}$. Thus $G^* \cdot \mathbf{N}^* > 0$ in all cases and (3.34) holds. The argument following (3.34) can be now repeated to rule out the possibility of exiting through $\text{arc}(u^5, u^{2\prime})$.

Consider now exit through $u^5$ ($u^{2\prime}$ can be treated similarly). By Lemma 3.3, the line $[P(u), Q(u)]$ is asymptotically normal to $\partial D$ at $Q_5$. This implies that $Q_{\leftarrow}(u) - Q(u)$ and $Q_{\rightarrow}(u) - Q(u)$ vanish as $u \to u^5$ along the curve $\text{arc}(u^4, u^5)$. Thus the right hand side of (3.24) vanishes in this limit, and it follows that $\partial\mathcal{L}$ is $C^1$ at $u^5$, with the unit inward normal $(0, -1)$ at this point. As in the preceding paragraph, Assumption 2.1 implies (3.36). The analysis of this case can now be finished by the same argument as in the case of an exit thorough $\text{arc}(u^5, u^{2\prime})$.

Consider now the possibility that $U$ exits through $\text{arc}(u^3, u^4)$, excluding $u^3$ and $u^4$. Recall $\theta$ defined in (3.7) and set $\theta^* = \theta(\tau)$. Then $\theta^* = \alpha$. If the trajectory of $U$ exits $\mathcal{L}$ at time $\tau$, then the trajectory of $\theta$ exits $[\alpha, \pi - \alpha]$ at the same time. Thus for every $\varepsilon > 0$ there exist $s, t$ such that $\tau \le s < t < \tau + \varepsilon$ and

$$(3.37)\qquad \theta(r) < \theta(s) = \alpha, \quad \text{for all } r \in (s, t].$$

Recall from (3.10) that $d\theta = V^{-1}\mathbf{p} \cdot (dM - dL)$. By Assumption 2.2, there is no upper right or lower left hinge within $(s, t)$, if $\varepsilon$ is small enough. This means a right hinge is necessarily lower. Thus, within this time interval,

$$\mathbf{p} \cdot dM = [\mathbf{p} \cdot \mathbf{n}(Y)]\,\mathrm{d}|M| \quad \text{and} \quad \mathbf{p} \cdot \mathbf{n}(Y) \ge 0.$$

Similarly, $\mathbf{p} \cdot \mathbf{n}(X) \le 0$. As a result, $\theta(t) \ge \theta(s)$, contradicting (3.37). We see that $U$ cannot exit $\mathcal{L}$ through $\text{arc}(u^3/u^4)$.

The discussion of the possible exit through $u^4$ will be split into two steps—one similar to the treatment of $\text{arc}(u^4, u^5)$ and the second one similar to that of $\text{arc}(u^3, u^4)$. One can show that the interior angle formed by $\partial\mathcal{L}$ at $u^4$ is less than or equal to $\pi$ but the calculation will not be provided here. If the angle is greater than $\pi$, then the first step of the argument given below would alone suffice to complete the proof.

Let us thus review the argument provided for $\text{arc}(u^4, u^5)$. Note that, by Lemma 3.3, the formula (3.28)–(3.29) for the one-sided tangent line to this arc is still valid for $u^* = u^4$, and that $\mathbf{n}(Q^*_{\rightharpoonup}) = i\mathbf{p}^*$ in this case. Consider a closed set $\mathcal{L}'$ whose boundary is the same as that of $\mathcal{L}$, except that the arc joining $u^3$ and $u^4$ is replaced by a curve that, in the vicinity of $u^4$, coincides with a ray starting from $u^4$ and is oriented as $-\mathbf{t}^*$. Note that $\partial\mathcal{L}'$ is $C^1$ at $u^4$, and, as before, $\mathbf{N}^* = (\mathbf{t}^*_2, -\mathbf{t}^*_1)$ is an inward normal to $\mathcal{L}'$. Recall that we have assumed that $U(t_k) \notin \mathcal{L}$ for a sequence $t_k \to \tau+$. We begin by showing that $U(t) \in \mathcal{L}'$ for all $t \in (\tau, \tau+\varepsilon)$, if $\varepsilon > 0$ is small enough. As in the argument for $\operatorname{arc}(u^4/u^5)$, we can achieve that by showing that $F^* \cdot \mathbf{N}^* > 0$ and $G^* \cdot \mathbf{N}^* > 0$. The argument leading to (3.31) holds. The one that leads to (3.33) is not valid since $\mathbf{p}^* \cdot \mathbf{n}(Q^*_{\rightharpoonup}) = 0$ and (3.32) can not be used. To obtain (3.33), note that $(P^* - Q^*) \cdot \mathbf{n}(Q^*_{\rightharpoonup}) = 0$, and, therefore, the left hand side of (3.33) can be written as

$$[(Q^*_{\rightharpoonup} - P^*) \cdot \mathbf{n}(Q^*_{\rightharpoonup})][(Q^* - P^*) \cdot \mathbf{n}(X^*)].$$

We have $(Q^*_{\rightharpoonup} - P^*) \cdot \mathbf{n}(Q^*_{\rightharpoonup}) < 0$, and, since $X^*$ has an upper left hinge,

$$(Q^* - P^*) \cdot \mathbf{n}(X^*) < 0.$$

Thus (3.33) holds. The argument following (3.33) can be repeated and one concludes that $U(t) \in \mathcal{L}'$ for all $t \in [\tau, \tau+\varepsilon)$, where $\varepsilon > 0$ is sufficiently small.

Next, note that $\mathbf{t}^* \in \mathbb{R}^2_+$. Hence if $B = B(u^4, \rho)$ denotes a disc and $C = B \cap \{(u_1, u_2) \in \mathbb{R}^2 : u_1 < u_1^4,\ u_2 < u_2^4\}$, then for sufficiently small $\rho > 0$ we have $B \cap (\mathcal{L}' \setminus \mathcal{L}) \subset C$. As a result, if $[P, Q] = \varphi^{-1}(u)$ for any $u \in C$, then $P_3 < P < P_4$ and $Q_3 < Q < Q_4$. Moreover, given any point $u \in C$, one can find a point $u' \in \operatorname{arc}(u^3, u^4)$ such that $u_1' > u_1$ and $u_2' = u_2$. Then $\angle\varphi^{-1}(u) < \angle\varphi^{-1}(u')$. The angle for each such $u'$ equals $\alpha$, by construction of $\operatorname{arc}(u^3, u^4)$. Thus, by Assumption 2.2, an upper right hinge does not exist for $\varphi^{-1}(u)$. The argument that we used for $\operatorname{arc}(u^3, u^4)$ can now be adapted to show that $U$ cannot exit $\mathcal{L}$ through $u^4$.

The proof is analogous for the other parts of the boundary of $\mathcal{L}$. □

## 4. MULTIPLICITY OF THE SECOND EIGENVALUE

In this section we prove Theorem 2.6. The overall strategy of the proof is similar to that in [3]. We begin by reformulating our main tool, Theorem 3.4, in a convenient way.

First, given $(x, y) \in \bar{D} \times \bar{D}$, $x \neq y$, let $m(x, y)$ be the line of symmetry for $x$ and $y$ and let $\{P_{x,y}, Q_{x,y}\} = m(x, y) \cap \partial D$, with the convention that the second coordinate of $P_{x,y}$ is less than or equal to that of $Q_{x,y}$. Let $T^1 \subset \bar{D} \times \bar{D}$ denote the set of pairs $(x, y)$, $x \neq y$, for which $\varphi(P_{x,y}, Q_{x,y}) \in \mathcal{L}$. Let

$$T = \{(x, y) \in T^1 : \mathbf{e}_1 \cdot (y - x) > 0\}.$$

For $(x, y) \in T$, let $\mathbb{P}_{x,y}$ denote a probability measure under which $(X(t), Y(t))$ is a mirror coupling starting from $(x, y)$, and recall that $Y(t) = X(t)$ for all

$t \geq \zeta$. Let $\mathbb{E}_{x,y}$ denote the corresponding expectation. An alternative statement of Theorem 3.4 is that if $(x, y) \in T$, then $\mathbb{P}_{x,y}$-a.s., $(X(t), Y(t)) \in T$ for all $t < \zeta$.

Let $D_L$ denote the connected component of $D \setminus \ell(P_1, Q_6')$ not containing $Q_6$ in its closure, and similarly let $D_R$ denote the connected component of $D \setminus \ell(P_1', Q_6)$ not containing $Q_6'$ in its closure. Because $\mathfrak{L}$ is constructed as a subset of $\mathbf{U}$, it follows from Theorem 3.4 that, for $(x, y) \in T$, $\mathbb{P}_{x,y}$-a.s., for all $t < \zeta$, $\ell(t)$ does not intersect $D_L$ or $D_R$. Thus, for $(x, y) \in T$,

$$X(t) \notin \overline{D_R} \text{ and } Y(t) \notin \overline{D_L} \quad \text{for all } t < \zeta,\ \mathbb{P}_{x,y}\text{-a.s.}\,. \tag{4.1}$$

We will use the following well known probabilistic representation of solutions to the heat equation. Suppose that $f_0$ is a bounded function on $\bar{D}$. Let $f : [0, \infty) \times \bar{D} \to \mathbb{R}$ denote the solution to $\frac{1}{2}\Delta f = (\partial/\partial t) f$ with initial values $f(0, x) = f_0(x)$ and Neumann boundary conditions on $\partial D$. Then

$$f(t, x) = \mathbb{E}_x f_0(X(t)). \tag{4.2}$$

In particular, if $\mu_2 > 0$ is the second eigenvalue for the Laplacian with Neumann boundary conditions and $\psi$ is any eigenfunction corresponding to $\mu_2$, then the above formula may be applied to $f(t, x) = e^{-\mu_2 t}\psi(x)$ and we obtain

$$\psi(x) = e^{\mu_2 t}\mathbb{E}_x f_0(X(t)). \tag{4.3}$$

***Lemma 4.1.*** *There exist constants $c_1$, $p_1 > 0$ such that for every $(x, y) \in T$,*

$$\mathbb{P}_{x,y}\Big(||X(1) - Y(1)|| \geq c_1 \mid \zeta > 1\Big) \geq p_1.$$

*Proof.* The assertion is the same as in Lemma 4 of [3]. The proof is very similar to that in [3] with minor, obvious adaptations, and is thus omitted. □

Let

$$S = \{f \in C(\bar{D}) : f(y) - f(x) \geq 0 \text{ for all } (x, y) \in T\},$$

$$\tilde{S} = \{f \in C(\bar{D}) : f(y) - f(x) > 0 \text{ for all } (x, y) \in T\}.$$

***Lemma 4.2.*** *If $\psi$ is a second Neumann eigenfunction and $\psi \in S$, then $\psi \in \tilde{S}$.*

*Proof.* Consider a second Neumann eigenfunction $\psi$ and assume that $\psi \in S$. Given $(x, y) \in T$, we shall show that $\psi(y) > \psi(x)$. Let us begin with $(x, y) \in T^o$ (the interior of $T$). Let $\varepsilon > 0$ be so small that $B(x, \varepsilon) \times B(y, \varepsilon) \subset T^o$. Since $\psi(x') \leq \psi(y')$ for $(x', y') \in T$ and $\psi$ is a non-constant real analytic function on $D$, there must exist $(x', y') \in B(x,\varepsilon) \times B(y,\varepsilon)$ where $\psi(x') < \psi(y')$. Thus there also exist balls $B_1 \subset B(x,\varepsilon)$, $B_2 \subset B(y,\varepsilon)$ and $\delta > 0$ such that

$$\psi(x') + \delta < \psi(y') \quad \text{for all } (x', y') \in B_1 \times B_2.$$

Consider a coupling of processes $(\tilde{X}(t), \tilde{Y}(t))$, in which $\tilde{X}(t)$ and $\tilde{Y}(t)$ are independent Brownian motions starting from $x$ and $y$, resp., until

$$\tau := \inf\Big\{t > 0 \mid (\tilde{X}(t),\ \tilde{Y}(t)) \in \partial(B(x,\varepsilon) \times B(y,\varepsilon))\Big\},$$

at which time they switch to a mirror coupling. Clearly $\tilde{X}(t)$ and $\tilde{Y}(t)$ are reflected Brownian motions in $\bar{D}$, starting from $x$ and $y$. By Theorem 3.4 and the strong Markov property applied at $\tau$, the process $(\tilde{X}(t), \tilde{Y}(t))$ does not leave the set $T$ for $t < \zeta$, a.s. . Thus, using (4.3), we obtain

$$e^{-\mu_2}(\psi(y) - \psi(x)) = \mathbb{E}_{x,y}(\psi(\tilde{Y}(1)) - \psi(\tilde{X}(1))) \geq \delta \mathbb{P}_{x,y}(H),$$

where $H$ denotes the event that $\tilde{X}$ and, respectively, $\tilde{Y}$ do not leave $B(x,\varepsilon)$ and $B(y,\varepsilon)$ before time 1, and $\tilde{X}(1) \in B_1$, $\tilde{Y}(1) \in B_2$. By well known properties of the standard Brownian motion, the probability of $H$ is strictly positive. This shows that $\psi(y) > \psi(x)$ for all $(x,y) \in T^{\circ}$.

To complete the proof, it suffices to show that for every $(x,y) \in T$, a mirror coupling $(X,Y)$ starting from $(x,y)$, reaches the interior of $T$ by time 1, and $\zeta > 1$, with positive probability. To this end it suffices to show that the process $U$, if it starts on $\partial\mathcal{L}$, enters the interior of $\mathcal{L}$ before time 1, and $\zeta > 1$, with positive probability. We analyze different parts of $\partial\mathcal{L}$ separately. If $U(0) \in \operatorname{arc}(u^3, u^4)$, consider $z \in \partial D$ and let $z'$ be the mirror image of $z$ with respect to $[P,Q] = \varphi^{-1}(U(0))$. Choose $z$ so that it has an upper left hinge and is located so close to $Q$ that for some $\varepsilon \in (0, \|z - z'\|/2)$ we have $B(z',\varepsilon) \subset D$. Let $D'$ be the connected component of $D \setminus [P,Q]$ that is on the left of $[P,Q]$. Consider the following event,

$$\Big\{X(t) \in D' \text{ for } t \in [0, \tfrac{1}{2}];\ X(t) \in B(z,\varepsilon) \text{ for } t \in (\tfrac{1}{2}, 1];$$
$$Y(t) \in B(z',\varepsilon) \text{ for } t \in (\tfrac{1}{2}, 1];\ |L|([0,1]) > 0\Big\}.$$

It is standard to prove that the above event has a strictly positive probability. Since $B(z',\varepsilon) \subset D$, we have $\zeta > 1$ if this event occurs. Since $\mathbf{p} \cdot \mathbf{n}(z) < 0$, it easily follows from (3.10) that $\theta(1) > \theta(0) = \alpha$. Thus the trajectory $U$ enters $\mathcal{L}^{\circ}$ if this event holds.

A similar argument applies for $U(0) \in \operatorname{arc}(u^4, u^5)$ with $z$ being a point on the boundary, close enough to $P$, having a lower right hinge (by Assumption 2.3 there is no lower left hinge for $[P,Q] \in \mathcal{A}(Q_4, Q_6)$ hence a lower right hinge must exist). Here one uses equations (3.8)–(3.9) to show that $U$ enters $\mathcal{L}^{\circ}$. For $U(0) \in \operatorname{arc}(u^5, u^{2\prime})$, take $z$ to be any boundary point to the right of $(P, Q)$ and use again (3.8) and (3.9).

Finally, consider the special boundary points $u^4$ and $u^5$. Now that it has been shown that the interior is reached from anywhere in $\partial\mathcal{L}$ save these special points, it suffices to show that the mirror line $\ell(t)$ simply moves (with positive probability) if it starts at the corresponding positions. However, the only way it can happen that the mirror does not move with probability 1 is when the domain $D$ is symmetric with respect to $\ell(0)$. This is clearly not the case for either $u^4$ or $u^5$, due to our assumptions. □

The following lemma essentially follows from Lemma 4.1 of [1], except that it has slightly weaker smoothness assumptions. The proof given here is shorter than that in [1].

***Lemma 4.3.*** *If $\psi$ is a Laplacian eigenfunction with Neumann boundary conditions corresponding to $\mu_2$ in a convex bounded domain $D$, then*

$$\sup_{x \in D} \|\nabla \psi(x)\| < \infty.$$

*Proof.* Consider any points $x, y \in \bar{D}$ and let $(X, Y)$ be a mirror coupling of reflected Brownian motions in $D$. Recall that $\zeta$ stands for the coupling time of $X$ and $Y$. By (4.3),

$$\left|\psi(y) - \psi(x)\right| = e^{\mu_2} \left| \mathbb{E}_{x,y}(\psi(Y(1)) - \psi(X(1))) \right| \le e^{\mu_2} \|\psi\|_\infty \mathbb{P}_{x,y}(\zeta > 1).$$

Since $D$ is a convex domain, $\|\psi\|_\infty < \infty$ (see, e.g., [4]). An application of the Itô formula and equations (3.1)–(3.3) show that

$$\|X - Y\| = \|x - y\| + \overline{W} + \bar{V},$$

where $\overline{W} = -2\int_0^{\cdot} m \cdot \mathrm{d}W$ and $\bar{V} = \int_0^{\cdot} (n(Y) \cdot m \, \mathrm{d}|M| - n(X) \cdot m \, \mathrm{d}|L|)$. The process $\overline{W}$ is a one dimensional Brownian motion (with the diffusion constant different from the standard one) and, by convexity of the domain, the process $\bar{V}$ is non-increasing. Hence,

$$\mathbb{P}_{x,y}(\zeta > 1) \le \mathbb{P}\Big(\inf_{0 \le t \le 1} \overline{W}_t > -\|x - y\|\Big) \le c_1 \|x - y\|.$$

We see that, for some $c_2 < \infty$,

$$\left|\psi(y) - \psi(x)\right| \le e^{\mu_2} ||\psi||_\infty c_1 ||x - y|| \le c_2 ||x - y||.$$

The lemma follows easily from this bound. □

For $\varepsilon > 0$, let

$$T_\varepsilon = \{(x, y) \in T : \|x - y\| \geq \varepsilon\}.$$

***Lemma 4.4.*** *Let $c_1$ be as in Lemma 4.1. For every $\varepsilon_1 \in (0, c_1)$ such that the interior of $T_{\varepsilon_1}$ is non-empty and every $\delta$, $\kappa > 0$ there exists $\varepsilon_2 > 0$ with the following property. If $\psi$ is a second Neumann eigenfunction satisfying*

$$\psi(y) - \psi(x) \geq \delta \quad \text{for all } (x, y) \in T_{\varepsilon_1}, \tag{4.4}$$

$$\psi(y) - \psi(x) \geq 0 \quad \text{for all } (x, y) \in T_{\varepsilon_2}, \tag{4.5}$$

*and*

$$\|\nabla \psi\| \leq \kappa \quad \text{on } D, \tag{4.6}$$

*then $\psi \in S$.*

*Proof.* Let $c_1$ and $p_1$ be as in the statement of Lemma 4.1. Fix any $\varepsilon_1 \in (0, c_1)$ such that the interior of $T_{\varepsilon_1}$ is non-empty and consider any $\delta$, $\kappa > 0$. Let

$$p_2 = \inf_{(x', y') \in T_{c_1}} \mathbb{P}_{x', y'}\Big((X(1), Y(1)) \in T_{\varepsilon_1}\Big). \tag{4.7}$$

It follows easily from Lemma 2 of [3] that $p_2 > 0$. Set $\varepsilon_2 = \min(\kappa^{-1}\delta p_1 p_2, \varepsilon_1)$ and note that $\varepsilon_2 > 0$. Let $\psi$ be a second Neumann eigenvalue satisfying (4.4)–(4.6). By (4.3),

$$e^{-\mu_2 t}(\psi(y) - \psi(x)) = \mathbb{E}_{x,y}\Big[\psi(Y(t)) - \psi(X(t))\Big], \quad t \geq 0.$$

Thus, it suffices to show that, for all $(x, y) \in T$,

$$\mathbb{E}_{x,y}[\psi(Y(2)) - \psi(X(2)) \mid \zeta > 1] \geq 0. \tag{4.8}$$

To this end, note that, in view of (4.6),

$$\begin{aligned}
&\mathbb{E}_{x,y}\Big[\psi(Y(2)) - \psi(X(2)) \mid \zeta > 1\Big] \\
&\qquad \geq \mathbb{E}_{x,y}\Big[(\psi(Y(2)) - \psi(X(2)))1_{\{\|X(2)-Y(2)\| \geq \varepsilon_2\}} \mid \zeta > 1\Big] - \kappa\varepsilon_2.
\end{aligned}$$

Since $(X(2), Y(2)) \in T$ a.s., the indicator function on the right hand side of the last formula can be replaced by $1_{\{(X(2),Y(2)) \in T_{\varepsilon_2}\}}$. By (4.5), the above inequality remains valid if the indicator function is further replaced by $1_{\{(X(2),Y(2)) \in T_{\varepsilon_1}\}}$. Thus by (4.4) and (4.7),

$$\begin{aligned}
&\mathbb{E}_{x,y}[\psi(Y(2)) - \psi(X(2)) \mid \zeta > 1] \\
&\qquad \geq \mathbb{E}_{x,y}[(\psi(Y(2)) - \psi(X(2)))1_{\{(X(2),Y(2)) \in T_{\varepsilon_1}\}} \mid \zeta > 1] - \kappa\varepsilon_2 \\
&\qquad \geq \delta \mathbb{P}_{x,y}((X(2), Y(2)) \in T_{\varepsilon_1} \mid \zeta > 1) - \kappa\varepsilon_2 \\
&\qquad \geq \delta \eta(x, y) p_2 - \kappa\varepsilon_2,
\end{aligned}$$

where

$$\eta(x,y) = \mathbb{P}_{x,y}\Big((X(1),Y(1)) \in T_{c_1} \mid \zeta > 1\Big).$$

By Lemma 4.1, $\eta(x,y) \geq p_1$. Thus

$$\mathbb{E}_{x,y}\Big[\psi(Y(2)) - \psi(X(2)) \mid \zeta > 1\Big] \geq \delta p_1 p_2 - \kappa\varepsilon_2 \geq 0,$$

and we have shown that (4.8) holds for all $(x,y) \in T$. This completes the proof of the lemma. ◻

*Proof of Theorem 2.6.* We start by showing that, whether the second Neumann eigenvalue is simple or not, there exists a corresponding eigenfunction that lies in $S$. The multiplicity of $\mu_2$ is either one or two, see [4, 11, 12]. Consider first the case when the multiplicity of $\mu_2$ is two and let $\psi$ and $\psi'$ be orthogonal Neumann eigenfunctions corresponding to $\mu_2$ and normalized so that $\int_D \psi^2 = \int_D (\psi')^2 = 1$. Since $\psi$ is real analytic in $D$, it is impossible that it vanishes on all of $D_R$; thus assume without loss of generality that it is strictly positive in some ball $B \subset D_R$. Let $f_0$ be a continuous nonnegative, nonzero function on $\bar{D}$, supported inside $B$. Let $f : [0,\infty) \times \bar{D} \to \mathbb{R}$ denote the solution to the heat equation in $\bar{D}$ with the initial values $f(0,x) = f_0(x)$ and Neumann boundary conditions on $\partial D$. The function $f$ has the following eigenfunction expansion:

$$f(t,x) = C_1 + (C_2\psi(x) + C_2'\psi'(x))e^{-\mu_2 t} + R(t,x), \tag{4.9}$$

where $C_1$, $C_2$ and $C_2'$ are suitable constants, and $\lim_{t\to\infty} e^{\mu_2 t}\sup_{x\in\bar{D}} |R(t,x)| = 0$ (see [4, Proposition 2.1]). Note that $C_2 = \int_D f_0\psi > 0$. Therefore $\psi_0 := C_2\psi + C_2'\psi'$ is a nonzero eigenfunction corresponding to $\mu_2$. We have by (4.9) for $(x,y) \in T$,

$$e^{\mu_2 t}(f(t,y) - f(t,x)) = \psi_0(y) - \psi_0(x) + \varepsilon(t,x,y)$$

where $\varepsilon(t,x,y) \to 0$ as $t \to \infty$. We now use (4.2) to write

$$\psi_0(y) - \psi_0(x) = e^{\mu_2 t}\mathbb{E}_{x,y}\Big[f_0(Y(t)) - f_0(X(t))\Big] - \varepsilon(t,x,y).$$

By (4.1) and the properties of $f_0$, it follows that $\psi_0(y) - \psi_0(x) \geq 0$ for all $(x,y) \in T$.

In the case when $\mu_2$ is simple, we take $\psi' \equiv 0$ and repeat the argument to conclude that $\psi \in S$. Obviously, if we assume that $\mu_2$ is simple, there is no logical need to prove any properties of eigenfunctions to finish the proof of Theorem 2.6. However, the fact that $\psi \in S$ is an interesting by-product of the proof.

In what follows, $\psi_0$ denotes an eigenfunction in $S$.

To prove that $\mu_2$ is simple, we use a variation of a proof from [3]. We argue by contradiction and assume that $\mu_2$ is not simple. Let $\psi_1$ denote a second Neumann eigenfunction orthogonal to $\psi_0$. It follows from Lemma 4.2 that $-\psi_1$ and $\psi_1$ cannot both lie in $S$; we thus assume without loss of generality that $\psi_1 \notin S$. Let

$$\begin{aligned} \psi_a &= (1-a)\psi_0 + a\psi_1, \quad 0 < a < 1, \\ a^* &= \inf\{a \in [0,1] : \psi_a \notin S\}. \end{aligned}$$

We claim that $a^* < 1$. If $a^* = 0$, then we are done. Otherwise, for $a < a^*$ and $(x,y) \in T$, one has $\psi_a(y) - \psi_a(x) \geq 0$. By continuity of the function $a \to \psi_a$, $\psi_{a^*}(y) - \psi_{a^*}(x) \geq 0$ for all $(x,y) \in T$. Thus

$$\psi_{a^*} \in S, \tag{4.10}$$

and, therefore, $a^* < 1$. This implies that

$$\exists\, a_k \downarrow a^*,\ a_k \in (a^*, 1), \quad \psi_{a_k} \notin S. \tag{4.11}$$

For $a = a_k$ as above, let

$$\varepsilon(a) = \sup\Big\{\|x-y\| : (x,y) \in T,\ \psi_a(y) - \psi_a(x) < 0\Big\}.$$

By (4.10) and Lemma 4.2, $\psi_{a^*} \in \tilde{S}$. When $k \to \infty$, $\psi_{a_k} \to \psi_{a^*}$ uniformly in $\bar{D}$. This easily implies that,

$$\varepsilon(a_k) \to 0 \quad \text{as } k \to \infty. \tag{4.12}$$

Fix some $\varepsilon_1 > 0$ as in Lemma 4.4. Since $\psi_{a^*} \in \tilde{S}$, we have $\psi_{a^*}(y) - \psi_{a^*}(x) > 0$ for all $(x,y)$ in the closed set $T_{\varepsilon_1}$. Thus there are constants $\delta > 0$ and $k_0$ such that for all $k > k_0$ and $(x,y) \in T_{\varepsilon_1}$, $\psi_{a_k}(y) - \psi_{a_k}(x) \geq \delta$. By Lemma 4.3, $\kappa := \sup_D(\|\nabla\psi_0\| + \|\nabla\psi_1\|) < \infty$. Let $\varepsilon_2 > 0$ be defined relative to $\varepsilon_1$, $\delta$ and $\kappa$ as in Lemma 4.4. By (4.12), we have that $\psi_{a_k}(y) - \psi_{a_k}(x) \geq 0$ for all $(x,y) \in T_{\varepsilon_2}$, provided that $k$ is large enough. Thus by Lemma 4.4, $\psi_{a_k} \in S$ for all large $k$. This contradicts (4.11). We conclude that $\mu_2$ is simple. ∎

***Corollary 4.5.*** *Let $D_M$ be the part of $D$ between $[P_3, Q_4']$ and $[P_3', Q_4]$. Suppose that the assumptions of Theorem 2.6 hold. Consider a second eigenfunction $\psi$ and define $\beta(x)$ by $\nabla\psi(x) = e^{i\beta(x)}$. Then $\beta(x) \in [\alpha - \pi/2, \pi/2 - \alpha]$ for all $x \in D_M$, or this assertion holds for $-\psi$.*

*Proof.* The corollary is an easy consequence of the fact that $\psi \in S$, established in the first part of the proof of Theorem 2.6. ∎

***Proposition 4.6.*** *Suppose that the second eigenvalue $\mu_2$ for the Laplacian with Neumann boundary conditions in a Lipschitz domain $D \subset \mathbb{R}^d$ is simple and there exist disjoint subsets $D', D'' \subset \bar{D}$ with non-empty interiors, non-empty open balls $B'$, $B'' \subset D$, and a coupling of reflected Brownian motions $(X, Y)$ in $D$ such that for any $X(0) = x \in B'$ and $Y(0) = y \in B''$ we have $X(t) \notin D'$ and $Y(t) \notin D''$ for all $t < \zeta := \inf\{t \geq 0 : X(t) = Y(t)\}$, a.s. . Let $\psi$ be an eigenfunction corresponding to $\mu_2$. Then $\psi(z) \geq 0$ for all $z \in D'$ and $\psi(z) \leq 0$ for all $z \in D''$, or this assertion applies to $-\psi$.*

*Proof.* Let $p_t(\cdot, \cdot)$ denote the transition density for the reflected Brownian motion in $D$. Consider any $x, y \in \bar{D}$. Then, by Proposition 2.1 of [4], for some $c_1 \in \mathbb{R}$, $c_2, c_3 \in (0, \infty)$, depending on $x$ and $y$,

$$p_t(x, z) - p_t(y, z) = c_1 e^{-\mu_2 t}\psi(z) + R(t, z), \tag{4.13}$$

and $|R(t, z)| \leq c_2 e^{-(\mu_2 + c_3)t}$, for all $t \geq 1$ and all $z \in D$. Recall that $\psi$ is a real analytic function that is not identically constant so it is not constant on balls $B'$ and $B''$. Hence, we can choose $x \in B'$ and $y \in B''$ such that $\psi(x) \neq \psi(y)$ and, therefore, $c_1 = \psi(x) - \psi(y) \neq 0$. Assume without loss of generality that $c_1 > 0$. Consider any non-empty open ball $B$ in the interior of $D'$. Then

$$\begin{aligned}
\int_B (p_t(x, z) - p_t(y, z))\,\mathrm{d}z &= \mathbb{P}_{x,y}(X(t) \in B) - \mathbb{P}_{x,y}(Y(t) \in B) \\
&= \mathbb{P}_{x,y}(X(t) \in B \mid t < \zeta) - \mathbb{P}_{x,y}(Y(t) \in B \mid t < \zeta) \\
&= -\mathbb{P}_{x,y}(Y(t) \in B \mid t < \zeta) \leq 0.
\end{aligned}$$

This shows that $p_t(x, \cdot) - p_t(y, \cdot)$ is non-positive in the interior of $D'$. We combine this with (4.13) and let $t \to \infty$ to see that $\psi(z) \leq 0$ for $z \in D'$. Similarly, $\psi(z) \geq 0$ for $z \in D''$. The inequalities are reversed if $c_1 < 0$. □

We note that the coupling of reflected Brownian motions in Proposition 4.6 is not assumed to be the mirror coupling. Among currently known couplings, the mirror coupling seems to be the only one which can satisfy the assumptions of Proposition 4.6. However, some new couplings are proposed from time to time (see, e.g., [2, 13]) so the proposition might be applied in the future to some other class of couplings.

***Corollary 4.7.*** *Suppose that $D$ is as in Theorem 2.6 and recall the definitions of $D_L$ and $D_R$ from the beginning of this section. The second eigenfunction $\psi$ for the Laplacian with Neumann boundary conditions in $D$ is non-negative on $D_L$ and non-positive on $D_R$, or this assertion applies to $-\psi$.*

*Proof.* The corollary follows from Proposition 4.6 and (4.1). □

The geometric location of the nodal line (i.e., zero set of the second eigenfunction) was studied in [2]. The results of that paper are logically independent from

Proposition 4.6 in the following sense. The techniques developed in [2] cannot be used to prove Corollary 4.7. On the other hand, the location of the nodal line in obtuse triangles is determined with greater accuracy in [2] than it could be done using Proposition 4.6.

We will next show how one can remove, in a sense, the assumptions of strict convexity and $C^2$-smoothness from Theorem 2.6. Suppose that $D \subset \mathbb{R}^2$ is bounded and convex but not necessarily strictly convex and $\partial D$ is not necessarily $C^2$-smooth. Suppose that $\{D_k\}_{k\geq 1}$ is a non-decreasing sequence of domains satisfying assumptions of Theorem 2.6 and converging to $D$, i.e., $\bigcup_{k\geq 1} D_k = D$. Let $P_1^k$ and $Q'_{6,k}$ be the points defined relative to $D_k$ and analogous to $P_1$ and $Q'_6$ in $\partial D$. Recall that these points are used to define the arc parametrization for parts of $\partial D_k$. Assume that there exist $P_1$ and $Q'_6$ such that $P_1^k \to P_1$ and $Q'_{6,k} \to Q'_6$ as $k \to \infty$. We make similar assumptions about existence of points $Q_1^k$ and $P'_{6,k}$ and their convergence to $Q_1$ and $P'_6$. Let $D_L$, $D_R \subset D$ be defined as at the beginning of this section, in terms of $P_1$, $Q'_6$, $P'_1$, and $Q_6$ described above. Let $\mathcal{L}_k$ be the Lyapunov set corresponding to the domain $D_k$ and let $\mathcal{L} = \overline{\limsup_{k\to\infty} \mathcal{L}_k}$, i.e., $\mathcal{L} = \overline{\bigcap_{n\geq 1} \bigcup_{k\geq n} \mathcal{L}_k}$. Let $T$ be defined as at the beginning of this section, relative to the present definition of $\mathcal{L}$.

***Proposition 4.8.*** *Assume that the above conditions for $D$ hold and every line of symmetry for $D$ is horizontal or vertical (hence $D$ may have two, one or no lines of symmetry). Assume that $D$ is not a rectangle, and that $D_L$, $D_R$ and $T$ have non-empty interiors. Then the assertion of Theorem 2.6 holds, i.e., the second eigenvalue for the Laplacian with Neumann boundary conditions in $D$ is simple.*

*Proof.* First we will prove that there exists a mirror coupling of reflected Brownian motions $(X, Y)$ in $D$ for which $\mathcal{L}$ is a Lyapunov set, i.e., if $(x, y) \in T$ and $(X(0), Y(0)) = (x, y)$, then $(X(t), Y(t)) \in T$ a.s., for all $t < \zeta := \inf\{t \geq 0 : X(t) = Y(t)\}$.

Fix any $(x, y) \in T$. It follows from the definition of $\mathcal{L}$ and $T$ that there exist $x_k$, $y_k \in D_k$ such that $x_k \to x$, $y_k \to y$ and $(x_k, y_k) \in T_k$, where $T_k$ is defined relative to $D_k$. For any $k$, let $(X_k, Y_k)$ be the mirror coupling of reflected Brownian motions in $D_k$ with $(X_k(0), Y_k(0)) = (x_k, y_k)$ as defined in (3.1)–(3.4). We construct all these processes on a single probability space and use the same process $W$ to define $X_k$, $Y_k$, $Z_k$, $\zeta_k$, and $\mathbf{m}_k$ for all $k$. By Theorem 2.3 of [8], $X_k$'s converge in distribution in the local uniform topology to a reflected Brownian motion in $D$ and an analogous statement is true for $Y_k$'s. In particular, there exists a filtration $(F_t)$ with respect to which the Brownian motion $W$ is a martingale, and there exist processes $X$, $Y$, $Z$, $L$ and $M$ such that $Z$ is a Brownian motion and an $(F_t)$-martingale, $X$, $Y$, $L$, and $M$ are $(F_t)$-adapted, and the equations (3.1)–(3.2) hold. Passing to a subsequence if necessary, $X_k$ converge to $X$ and $Y_k$ to $Y$ locally uniformly, with probability one. Let $\zeta = \inf\{t : \lim_{s\to t-}(X(s) - Y(s)) = 0\}$. By the uniform convergence result, for every $\delta > 0$ one has $\zeta_k \geq \zeta - \delta$ for all large

$k$. In particular, for all large $k$, $\mathbf{m}_k$ is well-defined for $t \le \zeta - \delta$, and $Z_k$ satisfies

$$Z_k(t) = W(t) - 2\int_0^t \mathbf{m}_k(s)\mathbf{m}_k(s) \cdot \mathrm{d}W(s)$$

for $t \le \zeta - \delta$. The processes $\mathbf{m}_k(\cdot \wedge (\zeta - \delta))$ converge uniformly to $\mathbf{m}(\cdot \wedge (\zeta - \delta))$. Let $I = \int_0^{\cdot} \mathbf{m}\mathbf{m} \cdot \mathrm{d}W$ and $I_k = \int_0^{\cdot} \mathbf{m}_k\mathbf{m}_k \cdot \mathrm{d}W$. Then

$$(I_k - I)(\cdot \wedge (\zeta - \delta)) = \int_0^{\cdot \wedge (\zeta - \delta)} (\mathbf{m}_k\mathbf{m}_k^T - \mathbf{m}\mathbf{m}^T)\,\mathrm{d}W,$$

and using Burkholder's inequality and the convergence of $\mathbf{m}_k$'s, the left hand side of the last formula converges locally uniformly to zero with probability one. Since $\delta > 0$ is arbitrary, we have shown that the equation (3.3) holds for $Z$ and $\mathbf{m}$ and all $t < \zeta$. We would like $(X, Y)$ to satisfy the definition of a mirror coupling given in Section 3 but at this point we do not know whether $Y = X$ on $[\zeta, \infty)$. Hence, we redefine $Y$ on $[\zeta, \infty)$ as $Y = X$ on this interval. The processes $X$, $Y$, $Z$, $L$, $M$, and $\mathbf{m}$ and the random variable $\zeta$ that we have constructed satisfy the definition of a mirror coupling in $D$, given in Section 3. As follows from [3], the process $(X(t), Y(t))$ is strong Markov. Lemma 4.1 applies to $(X, Y)$ because $(X, Y)$ is a mirror coupling in $D$. Also (4.7) holds for $(X, Y)$ with some $p_2 > 0$. It follows that the proof of Theorem 2.6 presented above applies in the present setting, except for Lemma 4.2, whose proof uses some properties of $\mathcal{L}$. Hence, it will suffice to prove that the assertion of Lemma 4.2 holds in the present context.

Part of the proof of Lemma 4.2 uses some explicit properties of $\mathcal{L}$. It might be possible to derive the needed properties of $\mathcal{L}$ from those of $\mathcal{L}_k$'s but that seems to be a hard and tedious task so we will use an alternative approach.

Consider a second Neumann eigenfunction $\psi$ and assume that $\psi \in S$. Consider any $(x, y) \in T$, $x \ne y$, and assume that $(X(0), Y(0)) = (x, y)$. We have by (4.3),

$$\psi(y) - \psi(x) = e^{\mu_2 t}\mathbb{E}_{x,y}\Big(\psi(Y(t)) - \psi(X(t))\Big).$$

Since $\psi(x') \le \psi(y')$ for $(x', y') \in T$ and $(X(t), Y(t)) \in T$ for all $t < \zeta$, the right hand side is non-negative. Moreover, the right hand side is strictly positive if for some $t \ge 0$ we have $\mathbb{P}_{x,y}(\psi(Y(t)) > \psi(X(t))) > 0$. Hence, it remains to consider only the case when $\mathbb{P}_{x,y}(\psi(Y(t)) > \psi(X(t))) = 0$ for every $t$. By continuity of $X$ and $Y$, this is equivalent to

$$\mathbb{P}_{x,y}\Big(\forall\, t,\ \psi(Y(t)) = \psi(X(t))\Big) = 1. \tag{4.14}$$

We have assumed that $x \ne y$ so $\zeta > 0$, a.s. . Reflected Brownian motion spends zero time on the boundary of $D$. Hence, there exists a (random) time interval $[t_1, t_2]$ with $t_1 < t_2 < \zeta$ such that $X(t), Y(t) \in D$ for all $t \in [t_1, t_2]$. It follows that the mirror $\ell(t)$ does not move during this time interval. Since $\psi$ is a real analytic function, we conclude from this and (4.14) that $\psi$ is symmetric with respect to $\ell^* := \ell(t_1)$.

Suppose that $D$ is symmetric with respect to $\ell^*$. Then $\ell^*$ is either vertical or horizontal, by the assumption made in the proposition. If $\ell^*$ is horizontal, then $x$ and $y$ lie on a vertical line but this is ruled out by the geometric assumptions on $D_k$'s. If $\ell^*$ is vertical, then $x$ and $y$ are symmetric with respect to the vertical line of symmetry of $D$. Then $D_L$ and $D_R$ are also symmetric and it is easy to see that the first part of the proof of Lemma 4.2 applies and one can conclude that $\psi(x) < \psi(y)$.

Next suppose that $D$ is not symmetric with respect to $\ell^*$. Then there is a positive probability that one and only one of the processes will spend some positive amount of local time on the boundary of $D$. This will move the mirror before time $\zeta$ and the same argument as before shows that there exists $\ell^{**} \neq \ell^*$ that is a line of symmetry for $\psi$. Moreover, $\ell^{**}$ can be chosen arbitrarily close to $\ell^*$. This easily implies that either $\psi$ is constant, which is impossible, or it is a function of only one variable in some orthonormal coordinate system. An argument given in the proof of Lemma 5 in [3] shows that $D$ must be a rectangle. We have assumed that $D$ is not a rectangle, so the proof of the proposition is complete. □

We believe that the assumptions on the domains $D_k$ converging to $D$ eliminate the possibility that $D$ has a line of symmetry that is not horizontal or vertical, or that $D$ is a rectangle, but proving this does not seem to be useful. Hence, we added an appropriate assumption about the shape of $D$ into Proposition 4.8.

## 5. EXAMPLES

Most of the assumptions on $D$ listed in Section 2 must be checked directly in concrete examples; doing so is a straightforward although tedious task. However, we would like to comment on Assumption 2.4. For any point $P \in \partial D$ with $P_1 < P < P_3$, one can find $[P, Q] \in \mathcal{A}(P_1, P_3)$ with $\angle(P, Q)$ arbitrarily close to $\angle(P)$, by the definition of $\mathcal{A}(P_1, P_3)$. It is clear, therefore, that Assumption 2.4 can be satisfied only if the curvature of $\partial D$ is decreasing in a neighborhood of $P$. Vice versa, if the curvature of $\partial D$ is non-increasing between $P_1$ and $P_3$, then the assumption is satisfied for $[P, Q] \in \mathcal{A}(P_1, P_3)$ with $\angle(P, Q)$ very close to $\angle(P)$ or "moderately" close to $\angle(P)$. For larger angles, Assumption 2.4 has to be verified directly.

***Example 5.1.*** We will analyze the domain $D$ depicted in Figure 5.1. The following conditions uniquely define the domain.

(1) The domain $D$ is convex and its boundary passes through points $(-2\sqrt{2}, 0)$, $(0, 2)$, $(3, 0)$, and $(0, -\sqrt{2})$.
(2) The boundary is a piece of an ellipse between points $(0, 2)$ and $(3, 0)$, with horizontal and vertical tangents at endpoints.

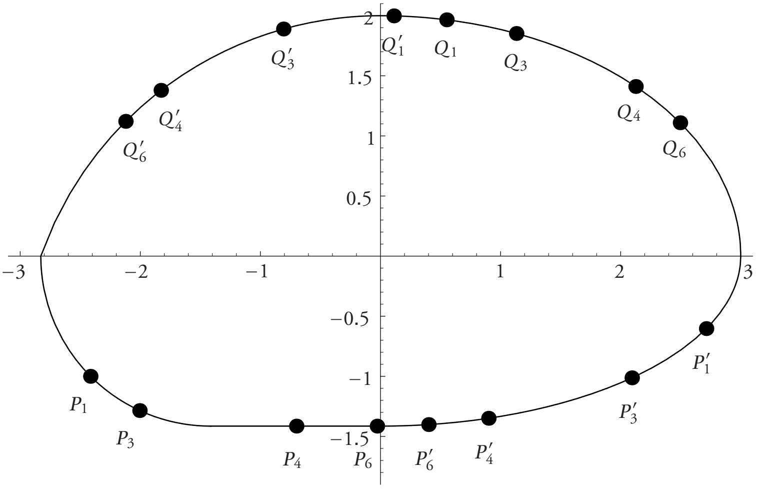



FIGURE 5.1. A domain with simple second Neumann eigenvalue.

(3) The boundary is a piece of an ellipse between points $(3, 0)$ and $(0, -\sqrt{2})$, with horizontal and vertical tangents at endpoints.
(4) The boundary is a circular arc with center at $(0, -1)$ and endpoints $(-2\sqrt{2}, 0)$ and $(0, 2)$. Note that the tangent line is horizontal at $(0, 2)$ but it is not vertical at $(-2\sqrt{2}, 0)$.
(5) The boundary is a piece of circular arc with center at $(-\sqrt{2}, 0)$ and endpoints $(-2\sqrt{2}, 0)$ and $(-\sqrt{2}, -\sqrt{2})$, with horizontal and vertical tangents at endpoints.
(6) The boundary is a horizontal line segment between points $(-\sqrt{2}, -\sqrt{2})$ and $(0, -\sqrt{2})$.

The domain $D$ is not strictly convex and it is not $C^1$. We will ignore these facts for the moment and we will proceed with a choice of parameters and special points as in Section 2. We take $\alpha = \pi/4$. This and the assumptions in Section 2 define uniquely points $P_1$, $P_3$, $P_4$, $P_6$, $Q_1$, $Q_3$, $Q_4$, $Q_6$, and the analogous points with primes. We will now describe how these points can be identified. $P_1$ is the unique point with $\angle(P_1) = \pi/4$. $Q_1$ is the unique point on the boundary with $\angle(P_1, Q_1) = \pi/4$. $Q'_6$ is the point with $\angle(Q'_6) = 7\pi/4$ and $P'_6$ is defined by $\angle(P'_6, Q'_6) = 7\pi/4$. The line segment $[P_3, Q_3]$ is chosen so that $\angle(P_3, Q_3) = \pi/4$ and $[P_3, Q_3] \cap [P'_6, Q'_6]$ is the midpoint of $[P'_6, Q'_6]$. Similarly, $[P'_4, Q'_4]$ is chosen so that $\angle(P'_4, Q'_4) = 7\pi/4$ and $[P_1, Q_1] \cap [P'_4, Q'_4]$ is the midpoint of $[P_1, Q_1]$. Other points are defined in the analogous way.

Because of the way the domain in our example is defined, the coordinates of all special points are algebraic numbers and can be written as explicit formulas involving only square roots. Some of the formulas are very complicated so we give coordinates of the special points in the approximated decimal form. See also

Figure 5.1.

$$\begin{aligned}
P_1 &= (-2.41, -1.00), & P_3 &= (-2.005, -1.28), \\
P_4 &= (-0.7, -1.41), & P_6 &= (-0.027, -1.41), \\
Q_1 &= (0.55, 1.97), & Q_3 &= (1.13, 1.85), \\
Q_4 &= (2.13, 1.41), & Q_6 &= (2.50, 1.11), \\
P_1' &= (2.71, -0.6), & P_3' &= (2.09, -1.01), \\
P_4' &= (0.9, -1.35), & P_6' &= (0.4, -1.40), \\
Q_1' &= (0.11, 2), & Q_3' &= (-0.81, 1.89), \\
Q_4' &= (-1.83, 1.38), & Q_6' &= (-2.12, 1.12).
\end{aligned}$$

We comment now on why $[P_3, Q_3]$ has been chosen so that $[P_3, Q_3] \cap [P_6', Q_6']$ is the midpoint of $[P_6', Q_6']$. Note that if $\ell(P', Q')$ is parallel to $[P_3, Q_3]$ and $P' > P_3$, then there are no lower left hinges for $\ell(P', Q')$. On the other hand, if $P' = P$ and $Q' < Q_3$, then there exists a lower left hinge for $\ell(P', Q')$. This observation is the basis of verification of Assumptions 2.2 and 2.3.

As for other assumptions listed in Section 2, some of them are elementary but tedious to verify so we omit the formal proof. The ones that are least trivial have been discussed at the beginning of this section. Also, the assumptions of Proposition 4.8 regarding the domains $D_R$ and $D_L$ follow from similar properties for the approximating sequence of domains.

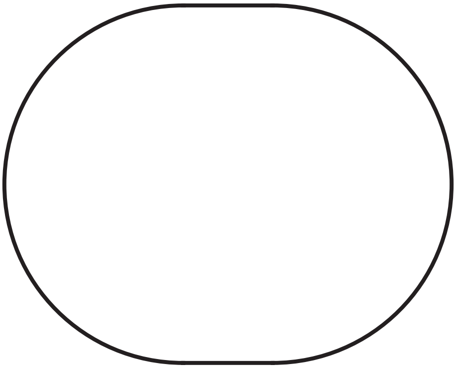


FIGURE 5.2. A domain with a line of symmetry and small diameter to width ratio.

Finally, note that because a part of $\partial D$ is a circular arc, Assumption 2.1 does not hold for some line segments such that $\angle(P, Q) = \angle(P)$. We have to address this as well as the fact that $D$ is not strictly convex and it is not $C^2$-smooth. Approximating the circular arc by that of an ellipse, it is easy to see that one can find a sequence of strictly convex $C^2$-smooth domains $D_k \uparrow D$, where Assumption

2.1 holds for each $D_k$. Moreover, $D_k$ can be chosen so that the points analogous to $P_j$'s, $Q_j$'s, $P_j'$'s and $Q_j'$'s and defined relative to $D_k$ converge to the analogous points in $D$. Hence, we can apply Proposition 4.8 and we conclude that the second Neumann eigenvalue in $D$ is simple. Corollary 4.7 implies that the second eigenfunction $\psi$ (or $-\psi$) is positive to the left of $[P_1, Q_6']$ and negative to the right of $[P_1', Q_6]$. By Corollary 4.5, $\angle(\nabla\psi(x)) \in [0, \pi/4] \cup (3\pi/4, \pi)$ for $x \in \bar{D}$ between $[P_3, Q_4']$ and $[P_3', Q_4]$.

***Example 5.2.*** Our next example is related to [10], [13] and an earlier article [4]. Jerison and Nadirashvili proved in [10] that the hot spots conjecture holds in all convex planar domains with two perpendicular axes of symmetry for all eigenfunctions corresponding to the second eigenvalue, but they left the question of the eigenvalue multiplicity open. Pascu proved in [13] that the hot spots conjecture holds for planar convex domains with a single line of symmetry, i.e., the maximum and minimum of the second Neumann eigenfunction are attained at the boundary of the domain. However, his theorem is stated for only one of many possible eigenfunctions corresponding to the second eigenvalue. The domain shown in Figure 5.2 has the boundary consisting of two circular arcs and two line segments. Since the ratio of its diameter to width is less than 1.53, Proposition 2.4 of [4] does not apply and we do not think that there is any other theorem in the literature that implies that the second Neumann eigenvalue is simple in this domain. This is indeed the case but we omit the detailed proof as it follows the lines of Example 5.1. We conclude that, in view of [13], the hot spots conjecture holds in its strongest form for the domain in Figure 5.2 and similar convex symmetric planar domains with at least one axis of symmetry.

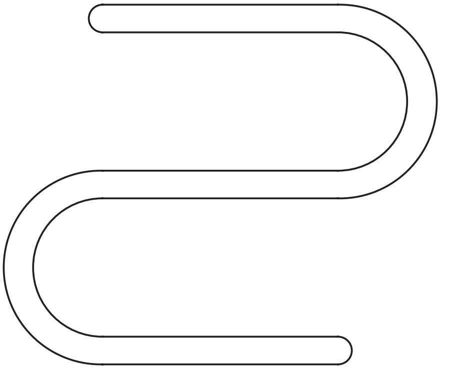


FIGURE 5.3. A snake domain.

***Example 5.3.*** We conclude with a challenge for the reader, similar in spirit to Exercise 4.1 in [4]. That exercise is concerned with a “snake” domain, i.e., a twisted version of a very thin “lip domain,” defined at the beginning of Section

1. Our present example is depicted in Figure 5.3. One can show that there exist subsets $D_L$ and $D_R$ of $D$ (close to the "endpoints" of $D$) and $T \subset \bar{D} \times \bar{D}$ such that (4.1) holds. Then an argument similar to that in the proof of Proposition 4.8 can be used to show that the second Neumann eigenvalue is simple in this domain. We leave the details of the proof as an exercise for the reader.

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