



On the multi-dimensional skew Brownian motion

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Abstract

We provide a new, concise proof of weak existence and uniqueness of solutions to the stochastic differential equation for the multidimensional skew Brownian motion. We also present an application to Brownian particles with skew-elastic collisions.

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1. Introduction

Let Σ denote the hyperplane $\{x \in \mathbb{R}^d : x_1 = 0\}$ in \mathbb{R}^d , $d \geq 1$, and let a vector field $b : \Sigma \rightarrow \mathbb{R}^d$ be given on it. Consider the stochastic differential equation (SDE), for a process X taking values in \mathbb{R}^d , of the form

$$X(t) = x + W(t) + \int_0^t b(X(s))dL(s), \quad t \geq 0, \tag{1.1}$$

where $x \in \mathbb{R}^d$, W is a standard d -dimensional Brownian motion, L is the symmetric local time of the first component of X at 0, b is bounded and Lipschitz, and writing $b = (b_1, \dots, b_d)'$,

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$b_1(x) \in [-1, 1]$ for all $x \in \Sigma$. This paper provides a new proof of weak existence and uniqueness of solutions to (1.1). A more general equation that allows for a bounded, measurable drift coefficient is also treated. These results are known by the work of Takanobu [19,18]. Our purpose here is to provide a much shorter proof. In the one-dimensional case this process is known as *skew Brownian motion*, first described by Itô and McKean [6], and further studied by Walsh [21] and treated by means of a SDE by Harrison and Shepp [5]. Moreover, the equation can be viewed as an extension of reflected Brownian motion. Indeed, if $b_1(x) = 1$ for all $x \in \Sigma$ then the equation describes a reflected Brownian motion in the half space with a reflection vector field given by b , a case first treated by Anderson and Orey [1]. See Lejay [9] for a comprehensive survey on skew Brownian motion and SDE involving local time.

The first to consider multidimensional skew Brownian motion is Portenko in [13–15]. The point of view of these papers is to consider a diffusion for which the drift term has a singularity localized at a (not necessarily flat) surface. This generalized drift coincides with the classical local time whenever the latter is well defined. Existence of solutions to the SDE is provided in [14]. Existence of weak solutions of stochastic differential equations with singular drift is now known in a very general setting. The paper [17] proves weak existence in a setting with unbounded and discontinuous drift given as a sum of countably many local times. This work in particular leads to a construction of a countably skew reflected Brownian motion with singular drift in bounded and unbounded multi-dimensional domains.

Zaitseva [24] studies an equation similar to (1.1) with b for which $b_1(x)$ does not depend on x and with an additional term $\sigma(X_s)d\tilde{B}(L(s))$, and obtains strong existence and uniqueness. In the special case $\sigma = 0$, the additional term alluded to above disappears, and one obtains precisely the form (1.1) with b_1 constant. In this case the first component of X is a one-dimensional skew Brownian motion and thus the questions of existence and uniqueness of solutions are considerably simpler than in the situation studied in the present paper. Additional works on multidimensional skew Brownian motion include Oshima [11], Trutnau [20] and Anulova [2]. The most relevant work for us is by Takanobu [19,18], where an equation of the form (1.1) with general drift and diffusion coefficients is treated. The first paper treats existence of weak solutions and the second proves uniqueness. Existence is shown by constructing a suitable approximating diffusion with a non-singular drift while uniqueness relies on rather involved arguments from Brownian excursion theory [22]. While the results of the present article are a special case of those established by Takanobu, the proof (of both existence and uniqueness) provided here is much shorter and more elementary.

Existence of solutions is established by constructing a ‘skew random walk’ and showing that in diffusion scaling it converges in distribution to a weak solution of the equation. The proof of weak uniqueness provided here is inspired by the technique used in [23] in a one-dimensional setting with time-varying skewness. Specifically, [23] considers the equation

$$X(t) = x + W(t) + \int_0^t \alpha(s)dL(s), \quad t \geq 0, \quad (1.2)$$

where $d = 1$ and $\alpha : [0, \infty) \rightarrow [-1, 1]$ is a measurable function. The key idea is to compute the conditional characteristic function $E(e^{i\lambda X(T)} \mid \mathcal{F}_t)$, for $0 \leq t \leq T < \infty$, $\lambda \in \mathbb{R}$ where $\mathcal{F}_t = \sigma\{X(s), W(s), L(s) : 0 \leq s \leq t\}$. In the setting of [23] this calculation reduces to characterizing the conditional law of $L(T) - L(t)$ given \mathcal{F}_t , which turns out to be simply the law of the local time (at time instant $T - t$) of a one-dimensional reflected Brownian motion. This is relevant to the multidimensional setting of (1.1) if one regards (1.2) as an equation for the first

component X_1 , and replaces the time-dependent skewness α by the state-dependent coefficient $b_1(X)$. Implementing this is, however, somewhat more involved than in [23]. In particular, the calculation of the conditional characteristic function requires a characterization of the *joint* law of $(X_2(T) - X_2(t), L(T) - L(t))$ given \mathcal{F}_t , where we write $X(t) = (X_1(t), X_2(t)) \in \mathbb{R} \times \mathbb{R}^{d-1}$. Lemmas 3.1 and 3.2 provide this characterization, by which the argument can be completed. We note that [23] in fact proves pathwise uniqueness. The proof of this result relies on the observation that if X and \tilde{X} are two solutions of (1.2) with associated local times L and \tilde{L} and common Brownian motion W then $X \vee \tilde{X}$ is also a solution, an idea that goes back to [8,12]. However, this approach fails in our setting.

The following notation will be used. For a Polish space \mathcal{S} , $\text{BM}(\mathcal{S})$ will denote the space of real bounded measurable functions on \mathcal{S} , $\mathcal{B}(\mathcal{S})$ the Borel σ -field on \mathcal{S} and $\mathcal{P}(\mathcal{S})$ will denote the space of probability measures on \mathcal{S} equipped with the topology of weak convergence. Also $\mathbb{C}_{\mathcal{S}} = C([0, \infty); \mathcal{S})$ (resp. $\mathbb{D}_{\mathcal{S}} = D([0, \infty); \mathcal{S})$) will denote the space of continuous functions (resp. RCLL functions) from $[0, \infty)$ to \mathcal{S} equipped with the usual local uniform topology (resp. Skorohod topology). A sequence of $\mathbb{D}_{\mathcal{S}}$ -valued random variables will be said to be C -tight if the corresponding sequence of probability laws on $\mathbb{D}_{\mathcal{S}}$ is relatively compact and any weak limit point of the sequence is supported on $\mathbb{C}_{\mathcal{S}}$. For $g \in \text{BM}(\mathcal{S})$, $\|g\|_{\infty} = \sup_{x \in \mathcal{S}} |g(x)|$. Elements in \mathbb{R}^d will be regarded as row vectors.

The rest of this paper is organized as follows. In Section 2 we state the main result and present some open problems related to it. The proof is provided in Section 3. Finally, Section 4 presents an application to a recent model of [4] for Brownian particles undergoing skew-elastic collisions.

2. Main result

Let $(\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_t\}_{t \geq 0})$ be a filtered probability space and let $\{W(t)\}_{0 \leq t < \infty}$ be a standard d -dimensional $\{\mathcal{F}_t\}$ -Brownian motion. Given a continuous $\{\mathcal{F}_t\}$ -semimartingale $\{Z(t)\}_{t \geq 0}$, define an $\{\mathcal{F}_t\}$ -adapted continuous process $\{L^Z(t)\}_{t \geq 0}$ as

$$L^Z(t) = |Z(t)| - |Z(0)| - \int_0^t \text{sgn}(Z(s)) dZ(s), \quad t \geq 0, \tag{2.1}$$

where for $x \in \mathbb{R}$, $\text{sgn}(x) = 1_{(0, \infty)}(x) - 1_{(-\infty, 0)}(x)$. L^Z is a nondecreasing process and is referred to as the symmetric local time of Z at 0 (see [16, Chapter VI]).

We consider a stochastic differential equation for a multidimensional process that involves its local time at the surface $\Sigma = \{x \in \mathbb{R}^d : x_1 = 0\}$. The equation takes the form

$$X(t) = x + W(t) + \int_0^t a(X(s)) ds + \int_0^t b(X(s)) dL(s), \quad t \geq 0, \tag{2.2}$$

where $X = (X_1, \dots, X_d)$, $x \in \mathbb{R}^d$, $a : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is bounded and measurable, $b : \Sigma \rightarrow \mathbb{R}^d$ is a bounded Lipschitz function satisfying $b_1(x) \in [-1, 1]$ for all $x \in \Sigma$ where b_1 denotes the first coordinate of b , and L is the symmetric local time of X_1 at 0. Note that L increases only at time instants t at which $X_1(t) = 0$. Thus in (2.2) the third integral is meaningful even though b is only defined on Σ . However to avoid confusion, for the rest of this work we will fix an arbitrary Lipschitz extension of b to all of \mathbb{R}^d and denote it once more as b . Our goal is to show that there exists a unique weak solution of (2.2). We recall below the definition of a weak solution. Let $\mathbb{C}_{\mathbb{R}^d} = C([0, \infty); \mathbb{R}^d)$, $\mathbb{C}_{\mathbb{R}} = C([0, \infty); \mathbb{R})$ and $\bar{\mathbb{C}} = \mathbb{C}_{\mathbb{R}^d} \times \mathbb{C}_{\mathbb{R}^d} \times \mathbb{C}_{\mathbb{R}}$. Denote by $\{\mathcal{G}_t\}_{t \geq 0}$ the

canonical filtration on $\bar{\mathbb{C}}$ and introduce canonical processes $\mathbf{x}, \mathbf{w}, \ell$ defined by

$$\begin{aligned} \mathbf{w}(t)(\bar{\omega}) &= \omega_1(t), & \mathbf{x}(t)(\bar{\omega}) &= \omega_2(t), \\ \ell(t)(\bar{\omega}) &= \omega_3, & \bar{\omega} &= (\omega_1, \omega_2, \omega_3) \in \bar{\mathbb{C}}, \quad t \geq 0. \end{aligned} \tag{2.3}$$

Definition 2.1. (a) A probability measure $P \in \mathcal{P}(\bar{\mathbb{C}})$ is said to be a weak solution of (2.2) if

(i) P -a.s.,

$$\mathbf{x}(t) = x + \mathbf{w}(t) + \int_0^t a(\mathbf{x}(s))ds + \int_0^t b(\mathbf{x}(s))d\ell(s), \quad t \geq 0.$$

(ii) Under P , \mathbf{w} is a $\{\mathcal{G}_t\}$ -Brownian motion.

(iii) Under P , ℓ is the symmetric local time of the first component of \mathbf{x} at 0.

(b) We say that (2.2) has a unique weak solution if whenever $P_1, P_2 \in \mathcal{P}(\bar{\mathbb{C}})$ are two weak solutions of (2.2), one has $P_1 = P_2$.

The following is our main result.

Theorem 2.1. *There exists a unique weak solution of (2.2).*

We end this section with a discussion of open problems addressing natural extensions of the above result.

Problem 1a. Determine whether pathwise uniqueness holds for (2.2).

An affirmative answer to the above might lead to strong existence by appealing to the well-known argument of Yamada and Watanabe.

Next, it is natural to consider a setting where Σ is replaced by a smooth surface of co-dimension 1, $\hat{\Sigma}$. In fact, this setting goes back to the aforementioned papers by Portenko. Let such a surface be given along with a vector field $b : \hat{\Sigma} \rightarrow \mathbb{R}^d$, and assume

$$b(x) \cdot n(x) \in [-1, 1] \quad \text{for every } x \in \hat{\Sigma},$$

where n denotes a unit normal to $\hat{\Sigma}$. Then Eq. (2.2), with L denoting the local time of X at $\hat{\Sigma}$, is indeed more general than the one treated in this paper (one defines local time here by means of limits of occupation time in the vicinity of the surface as e.g. in [3]). Note that this equation includes as a special case Brownian motion reflected on a smooth surface, i.e. when $b \cdot n = 1$ on $\hat{\Sigma}$, for which existence and uniqueness is well-understood [1,10]. One may address this equation by flattening $\hat{\Sigma}$ to Σ , that is, by applying a twice continuously differentiable, invertible map $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ that maps (locally) $\hat{\Sigma}$ into Σ . Setting $Y = \varphi(X)$ one obtains an equation for Y of the form

$$Y(t) = y + \int_0^t \sigma(Y(s))dW(s) + \int_0^t a(Y(s))ds + \int_0^t c(Y(s))dL(s), \tag{2.4}$$

where L denotes the local time of Y at Σ and the coefficients σ, a and c depend on b and on derivatives, up to second order, of φ . This translates the equation into one that involves the local time on a flat surface, and SDE with coefficients, and so weak existence and uniqueness are available by [19,18]. However, strong form of uniqueness is missing, which leads us to the following extended version of Problem 1a.

Problem 1b. Determine whether pathwise uniqueness holds for (2.4) for a smooth surface $\hat{\Sigma}$.

Finally, consider Eq. (2.2) in dimension 2 with discontinuous coefficient $b = (1, 0)'$ on $\{x_1 = 0, x_2 \leq 0\}$ and $b = (-1, 0)$ on $\{x_1 = 0, x_2 > 0\}$ (say, with $a = 0$). The authors are not aware of uniqueness results for this nonstandard, yet simple equation of a reflected Brownian motion. It would be interesting to understand this equation in the following broader context.

Problem 2. Determine whether any form of uniqueness holds for (2.2) beyond Lipschitz continuity of the coefficient b .

3. Proof

The proof has three parts. We begin by showing that the problem can be reduced to the case where $a = 0$. Namely, we show, in Section 3.1 that Theorem 2.1 follows from

Proposition 3.1. *Suppose that $a = 0$. Then (2.2) has a unique weak solution.*

Then we turn to proving Proposition 3.1. Setting $a = 0$, we prove uniqueness and then existence in Sections 3.2 and 3.3, respectively.

3.1. Reduction to the case $a = 0$

Theorem 2.1 follows from Proposition 3.1 by a straightforward application of Girsanov’s theorem, as we now show.

Proof of Theorem 2.1. Let P_0 be the unique solution of (2.2) with $a = 0$. Define $\tilde{P} \in \mathcal{P}(\bar{\mathbb{C}})$ as

$$\tilde{P}(A) = \int_A \mathcal{E}_T dP_0, \quad A \in \mathcal{G}_T, \quad T > 0,$$

where

$$\mathcal{E}_T = \exp \left\{ \int_0^T a(\mathbf{x}(s)) \cdot d\mathbf{w}(s) - \frac{1}{2} \int_0^T |a(\mathbf{x}(s))|^2 ds \right\}.$$

Let $\tilde{\mathbf{w}}(t) = \mathbf{w}(t) - \int_0^t a(\mathbf{x}(s)) ds$, $t \geq 0$ and let $P \in \mathcal{P}(\bar{\mathbb{C}})$ be defined as $P = \tilde{P} \circ (\tilde{\mathbf{w}}, \mathbf{x}, \ell)^{-1}$. Then by Girsanov’s theorem P is a weak solution of (2.2).

To see uniqueness of a weak solution, let P_1 be another weak solution of (2.2). Define

$$\tilde{P}_1(A) = \int_A \tilde{\mathcal{E}}_T dP_1, \quad A \in \mathcal{G}_T, \quad T > 0,$$

where

$$\tilde{\mathcal{E}}_T = \exp \left\{ - \int_0^T a(\mathbf{x}(s)) \cdot d\mathbf{w}(s) - \frac{1}{2} \int_0^T |a(\mathbf{x}(s))|^2 ds \right\}.$$

Let $\tilde{\mathbf{w}}_1(t) = \mathbf{w}(t) + \int_0^t a(\mathbf{x}(s)) ds$. Then $\tilde{P}_0 = \tilde{P}_1 \circ (\tilde{\mathbf{w}}_1, \mathbf{x}, \ell)^{-1}$ is a weak solution of (2.2) with $a = 0$ and so by Proposition 3.1 $\tilde{P}_0 = P_0$. Finally for all $T > 0$ and $f \in \text{BM}(\bar{\mathbb{C}})$

$$\begin{aligned} E_1 [f(\mathbf{w}(\cdot \wedge T), \mathbf{x}(\cdot \wedge T), \ell(\cdot \wedge T))] &= \tilde{E}_1 \left[f(\mathbf{w}(\cdot \wedge T), \mathbf{x}(\cdot \wedge T), \ell(\cdot \wedge T)) \tilde{\mathcal{E}}_T^{-1} \right] \\ &= \tilde{E}_0 \left[f \left(\mathbf{w}(\cdot \wedge T) - \int_0^{\cdot \wedge T} a(\mathbf{x}(s)) ds, \mathbf{x}(\cdot \wedge T), \ell(\cdot \wedge T) \right) \mathcal{E}_T \right] \end{aligned}$$

$$\begin{aligned}
 &= E_0 \left[f \left(\mathbf{w}(\cdot \wedge T) - \int_0^{\cdot \wedge T} a(\mathbf{x}(s)) ds, \mathbf{x}(\cdot \wedge T), \boldsymbol{\ell}(\cdot \wedge T) \right) \mathcal{E}_T \right] \\
 &= \tilde{E} \left[f \left(\mathbf{w}(\cdot \wedge T) - \int_0^{\cdot \wedge T} a(\mathbf{x}(s)) ds, \mathbf{x}(\cdot \wedge T), \boldsymbol{\ell}(\cdot \wedge T) \right) \right] \\
 &= E [f(\mathbf{w}(\cdot \wedge T), \mathbf{x}(\cdot \wedge T), \boldsymbol{\ell}(\cdot \wedge T))].
 \end{aligned}$$

Thus we have $P_1 = P$. \square

For the rest of the section we assume that $a = 0$.

3.2. Weak uniqueness

For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ write

$$y = (y_1, y_2) \in \mathbb{R}^d \quad \text{where } y_1 = |x_1|, \quad y_2 = (x_2, \dots, x_d).$$

Define $\beta : \mathbb{R}^{d-1} \rightarrow \mathbb{R}^d$ as $\beta(z) = b(0, z_1, \dots, z_{d-1})$, $z = (z_1, \dots, z_{d-1}) \in \mathbb{R}^{d-1}$. Write $\beta(z) = (\beta_1(z), \beta_2(z)) \in \mathbb{R} \times \mathbb{R}^{d-1}$, $z \in \mathbb{R}^d$. Consider the equations

$$S(t) = y_1 + B_1(t) + V(t), \tag{3.1}$$

$$Y(t) = y_2 + B(t) + \int_0^t \beta_2(Y(s)) dV(s). \tag{3.2}$$

Here B_1 and B are independent 1- and $(d - 1)$ -dimensional Brownian motions, respectively, S is a continuous nonnegative process and V is a continuous nondecreasing process with $V(0) = 0$, which increases only when $S = 0$.

The following lemma shows that (3.1)–(3.2) has a unique weak solution. Recall the coordinate processes $\mathbf{w}, \mathbf{x}, \boldsymbol{\ell}$ on $\tilde{\mathbb{C}}$ introduced in (2.3) and write $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$ and $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$, where \mathbf{w}_1 and \mathbf{x}_1 take values in \mathbb{R} while \mathbf{w}_2 and \mathbf{x}_2 take values in \mathbb{R}^{d-1} .

Lemma 3.1. (a) *There exists a $Q \in \mathcal{P}(\tilde{\mathbb{C}})$ such that*

(i) *Q -a.s., $\mathbf{x}_1(t) \geq 0$ for all t , and $t \mapsto \boldsymbol{\ell}(t)$ is nondecreasing and $\boldsymbol{\ell}(0) = 0$.*

(ii) *Q -a.s.,*

$$\mathbf{x}_1(t) = y_1 + \mathbf{w}_1(t) + \boldsymbol{\ell}(t),$$

$$\mathbf{x}_2(t) = y_2 + \mathbf{w}_2(t) + \int_0^t \beta_2(\mathbf{x}_2(s)) d\boldsymbol{\ell}(s), \quad t \geq 0.$$

(iii) *Under Q , \mathbf{w}_1 and \mathbf{w}_2 are independent 1- and $(d - 1)$ -dimensional $\{\mathcal{G}_t\}$ -Brownian motions.*

(iv) *Q -a.s., $\int_{[0, \infty)} 1_{(0, \infty)}(\mathbf{x}_1(t)) d\boldsymbol{\ell}(t) = 0$.*

(b) *Let Q_1 and Q_2 be two probability measures as in part (a). Then $Q_1 = Q_2$.*

Proof. The result is an immediate consequence of the following two observations:

- Given a 1-dimensional Brownian motion B_1 , there is a unique pair of processes (S, V) that satisfies (3.1) and is such that $S(t) \geq 0$, $V(0) = 0$, V is nondecreasing and $\int_{[0, \infty)} 1_{(0, \infty)}(S(t)) dV(t) = 0$. This unique pair is given explicitly as

$$\begin{aligned}
 S(t) &= y_1 + B_1(t) - \inf_{0 \leq s \leq t} \{y_1 + B_1(s)\} \wedge 0, \\
 V(t) &= (S(t) - y_1 - B_1(t)), \quad t \geq 0.
 \end{aligned} \tag{3.3}$$

- Given a $(d - 1)$ -dimensional Brownian motion B and V defined as above, we have, using the Lipschitz property of β and a standard argument based on Gronwall’s lemma, that there exists a unique continuous process Y that solves (3.2). \square

In order to indicate the dependence of Q in Lemma 3.1 on the initial condition $(y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R}^{d-1}$ we will denote this probability measure as Q_{y_1, y_2} . It is easy to see that $(y_1, y_2) \mapsto Q_{y_1, y_2}$ is a continuous function. Indeed, let $\tilde{B} = (B_1, \dots, B_d)'$ be a d dimensional Brownian motion given on some probability space and let $y^n \rightarrow y$ in \mathbb{R}^d . Define S^n, V^n as in (3.3) replacing y_1 with the first component y_1^n of y^n . Also define S, V by (3.3). Then clearly $(S^n, V^n) \rightarrow (S, V)$ a.s. in $\mathbb{C}_{\mathbb{R}^2}$. Next, letting $B = (B_2, \dots, B_d)'$, let Y^n be the unique solution of the equation

$$Y^n(t) = y_2^n + B(t) + \int_0^t \beta_2(Y^n(s))dV^n(s), \quad t \geq 0$$

where y_2^n is the $(d - 1)$ dimensional vector consisting of the last $d - 1$ components of y^n . Define Y similarly by replacing y_2^n with y_2 and V^n with V . Then for some $\kappa \in (0, \infty)$ and for all $T \geq 0$

$$\sup_{0 \leq t \leq T} |Y^n(t) - Y(t)| \leq \theta^n(T) + \kappa \int_0^T \sup_{0 \leq u \leq s} |Y^n(u) - Y(u)|dV^n(u),$$

where

$$\theta^n(T) = |y_2^n - y_2| + \sup_{0 \leq t \leq T} \left| \int_0^t \beta_2(Y(s))dV^n(s) - \int_0^t \beta_2(Y(s))dV(s) \right|.$$

An application of Gronwall’s lemma now gives

$$\sup_{0 \leq t \leq T} |Y^n(t) - Y(t)| \leq \theta^n(T)e^{\kappa V^n(T)}.$$

Since $y^n \rightarrow y$ and $V^n \rightarrow V$ a.s. in $\mathbb{C}_{\mathbb{R}}$ we have that the right side of the above display converges a.s. to 0 for every $T \geq 0$. Thus $Y^n \rightarrow Y$ a.s. in $\mathbb{C}_{\mathbb{R}}$. Finally note that $Q_{y_1^n, y_2^n}$ [resp. Q_{y_1, y_2}] is the probability law of $(\tilde{B}, (S^n, Y^n), V^n)$ [resp. $(\tilde{B}, (S, Y), V)$] proving that $Q_{y_1^n, y_2^n}$ converges to Q_{y_1, y_2} . This proves the continuity of the map $(y_1, y_2) \mapsto Q_{y_1, y_2}$.

Let P be a weak solution of (2.2) (recall that we are taking $a = 0$) and for $t \geq 0$ consider the following stochastic processes on $(\bar{\mathbb{C}}, \mathcal{B}(\bar{\mathbb{C}}), P)$

$$\begin{aligned} \hat{b}^t(\cdot) &= \int_t^{t+\cdot} \text{sgn}(x_1(s))d\mathbf{w}_1(s), & \hat{w}_2^t(\cdot) &= \mathbf{w}_2(t + \cdot) - \mathbf{w}_2(t), \\ \hat{x}_1^t(\cdot) &= \mathbf{x}_1(t + \cdot), & \hat{x}_2^t(\cdot) &= \mathbf{x}_2(t + \cdot), & \hat{\ell}^t(\cdot) &= \ell(t + \cdot) - \ell(t). \end{aligned}$$

Also let $\mathbf{z} = (\mathbf{w}, \mathbf{x}, \ell), \mathbf{z}^t(\cdot) = \mathbf{z}(\cdot \wedge t), \hat{\mathbf{z}}^t = (\hat{b}^t, \hat{w}_2^t, |\hat{x}_1^t|, \hat{x}_2^t, \hat{\ell}^t)$ and $\mathbf{u}^t = (\mathbf{z}^t, \hat{\mathbf{z}}^t)$. Note that, for each t, \mathbf{u}^t is a random variable with values in $\bar{\mathbb{C}} \times \bar{\mathbb{C}}$. Let $Q^t = P \circ (\mathbf{u}^t)^{-1}$ and disintegrate Q^t as

$$Q^t(d\bar{\omega}, d\hat{\omega}) = \mathcal{R}^t(d\bar{\omega})\mathcal{P}^t(\bar{\omega}, d\hat{\omega}), \quad (\bar{\omega}, \hat{\omega}) \in \bar{\mathbb{C}} \times \bar{\mathbb{C}}.$$

Note that \mathcal{R}^t is the probability law of \mathbf{z}^t under P and $\mathcal{P}^t(\mathbf{z}^t, \cdot)$ is the conditional law under P of $\hat{\mathbf{z}}^t$ given \mathbf{z}^t .

With the above notation we have the following lemma.

Lemma 3.2. Fix $t \geq 0$. Then for \mathcal{R}^t -a.e. $\bar{\omega}, \mathcal{P}^t(\bar{\omega}, d\hat{\omega}) = Q_{|\bar{x}_1(t)|, \bar{x}_2(t)}(d\hat{\omega})$, where we write $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3) \in \mathbb{C}_{\mathbb{R}^d} \times \mathbb{C}_{\mathbb{R}^d} \times \mathbb{C}_{\mathbb{R}}, (\bar{x}_1(t), \bar{x}_2(t)) = \bar{\omega}_2(t) \in \mathbb{R} \times \mathbb{R}^{d-1}$.

Proof. Using Tanaka’s formula (2.1) and the fact that $\hat{\ell}^t$ is the symmetric local time of \hat{x}_1^t , (as follows from Definition 2.1(iii)), we have

$$|\hat{x}_1^t(\cdot)| = |x_1(t)| + \int_0^\cdot \text{sgn}(\hat{x}_1^t(s))d\hat{x}_1^t(s) + \hat{\ell}^t(\cdot).$$

Using Lemma 3.1(a)(iv) and the property $\text{sgn}(0) = 0$ one has $\int_{[0,\infty)} \text{sgn}(x_1(s))d\ell(s) = 0$. Hence by the expression for x_1 from Definition 2.1(i), P -a.s.,

$$\begin{aligned} |\hat{x}_1^t(\cdot)| &= |x_1(t)| + \hat{b}^t(\cdot) + \hat{\ell}^t(\cdot), \\ \hat{x}_2^t(\cdot) &= x_2(t) + \hat{w}_2^t(\cdot) + \int_0^\cdot \beta_2(\hat{x}_2^t(s))d\hat{\ell}^t(s). \end{aligned}$$

In view of Lemma 3.1 it suffices to show that for each fixed $t \geq 0$, under P , conditionally on z^t , \hat{w}_2^t and \hat{b}^t are independent $(d - 1)$ -dimensional and 1-dimensional Brownian motions and are adapted to the filtration generated by \hat{z}^t , and have future increments independent of that filtration. Namely,

$$P \left[(\hat{b}^t, \hat{w}_2^t) \in A \mid \sigma(z^t) \right] = P(w \in A), \quad \text{for all } A \in \mathcal{B}(\mathbb{C}_{\mathbb{R}^d}),$$

for all $f \in \text{BM}(\mathbb{C}_{\mathbb{R}} \times \mathbb{C}_{\mathbb{R}^{d-1}})$ and $t_1 \geq 0$

$$E \left[f(\hat{b}^t(\cdot \wedge t_1), \hat{w}_2^t(\cdot \wedge t_1)) \mid \sigma(z^t) \vee \sigma(\hat{z}^t(s) : 0 \leq s \leq t_1) \right] = f(\hat{b}^t(\cdot \wedge t_1), \hat{w}_2^t(\cdot \wedge t_1)),$$

and for all $0 \leq t_1 \leq t_2 < \infty$, $g \in \text{BM}(\bar{\mathbb{C}})$, $h \in \text{BM}(\mathbb{R}^d)$

$$\begin{aligned} &E \left[g(\hat{z}^t(\cdot \wedge t_1))h(\hat{b}^t(t_2) - \hat{b}^t(t_1), \hat{w}^t(t_2) - \hat{w}^t(t_1)) \mid \sigma(z^t) \right] \\ &= E \left[g(\hat{z}^t(\cdot \wedge t_1)) \mid \sigma(z^t) \right] E \left[h(\hat{b}^t(t_2) - \hat{b}^t(t_1), \hat{w}^t(t_2) - \hat{w}^t(t_1)) \mid \sigma(z^t) \right]. \end{aligned}$$

The above properties are immediate from the definitions. \square

We can now prove the following.

Proposition 3.2. *With $a = 0$, there is at most one weak solution of (2.2).*

Proof. It suffices to show that for every $m \geq 1$ and $0 \leq t_1 \leq t_2 \leq \dots \leq t_m < \infty$ the probability law $P \circ (x(t_1), \dots, x(t_m))^{-1}$ on $(\mathbb{R}^d)^m$ is the same for any weak solution P of (2.2). For simplicity we only consider the case $m = 2$ and write $t_1 = t, t_2 = T$. Fix $\lambda \in \mathbb{R}^d$. Then $z(\cdot) = \exp(\iota\lambda \cdot x(\cdot))$ satisfies for all $0 \leq t \leq s$

$$z(s) = z(t) + \iota \int_t^s z(u)\lambda \cdot dw(u) - \frac{|\lambda|^2}{2} \int_t^s z(u)du + \iota \int_t^s z(u)\lambda \cdot \beta(x_2(u))d\ell(u).$$

Thus writing for $s \geq t$, $E(z(s) \mid \mathcal{G}_t) = G(s)$, we have

$$\begin{aligned} G(s) &= G(t) - \frac{|\lambda|^2}{2} \int_t^s G(u)du \\ &+ E \left[\iota \int_0^{s-t} \exp(\iota\lambda_2 \cdot \hat{x}_2^t(u))\lambda \cdot \beta(\hat{x}_2^t(u))d\hat{\ell}^t(u) \mid \mathcal{G}_t \right] \end{aligned} \tag{3.4}$$

where $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R} \times \mathbb{R}^{d-1}$. For $(y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R}^{d-1}$ denote the expectation under Q_{y_1, y_2} as E_{y_1, y_2}^Q and let

$$E_{y_1, y_2}^Q \left[\int_0^{s-t} \exp(t\lambda_2 \cdot \mathbf{x}_2(u)) \lambda \cdot \beta(\mathbf{x}_2(u)) d\ell(u) \right] = H^{t,s}(y_1, y_2).$$

Then from Lemma 3.2 and (3.4) we have for all $s \geq t$

$$G(s) = G(t) - \frac{|\lambda|^2}{2} \int_t^s G(u) du + H^{t,s}(|\mathbf{x}_1(t)|, \mathbf{x}_2(t)).$$

Thus we have shown that for all $0 \leq t \leq T < \infty$ and $\lambda \in \mathbb{R}^d$ there is a measurable function $H_\lambda^{t,T}$ from $\mathbb{R}_+ \times \mathbb{R}^{d-1}$ to the complex plane such that for every weak solution P of (2.2)

$$E \left[\exp(t\lambda \cdot \mathbf{x}(T)) \mid \mathcal{G}_t \right] = H_\lambda^{t,T}(|\mathbf{x}_1(t)|, \mathbf{x}_2(t)), \quad P\text{-a.s.}$$

By a standard approximation argument it now follows that for all $0 \leq t \leq T < \infty$ and for every $\varphi \in \text{BM}(\mathbb{R}^d)$ there is a function $H_\varphi^{t,T} \in \text{BM}(\mathbb{R}_+ \times \mathbb{R}^{d-1})$ such that for every weak solution P of (2.2)

$$E [\varphi(\mathbf{x}(T)) \mid \mathcal{G}_t] = H_\varphi^{t,T}(|\mathbf{x}_1(t)|, \mathbf{x}_2(t)), \quad P\text{-a.s.} \tag{3.5}$$

Finally let $\varphi_1, \varphi_2 \in \text{BM}(\mathbb{R}^d)$ and let P be a weak solution of (2.2). Then

$$\begin{aligned} E [\varphi_1(\mathbf{x}(t))\varphi_2(\mathbf{x}(T))] &= E [\varphi_1(\mathbf{x}(t))E [\varphi_2(\mathbf{x}(T)) \mid \mathcal{G}_t]] \\ &= E \left[\varphi_1(\mathbf{x}(t))H_{\varphi_2}^{t,T}(|\mathbf{x}_1(t)|, \mathbf{x}_2(t)) \right]. \end{aligned}$$

Note that

$$\psi(z) = \varphi_1(z)H_{\varphi_2}^{t,T}(|z_1|, z_2), \quad z = (z_1, z_2) \in \mathbb{R} \times \mathbb{R}^{d-1}$$

is in $\text{BM}(\mathbb{R}^d)$. Thus from (3.5)

$$E [\varphi_1(\mathbf{x}(t))\varphi_2(\mathbf{x}(T))] = H_\psi^{0,t}(|x_1|, x_2).$$

As a result, for every weak solution P of (2.2), the probability law $P \circ (\mathbf{x}(t), \mathbf{x}(T))^{-1}$ does not depend on P . The result follows. \square

3.3. Weak existence

Proposition 3.3. *With $a = 0$, there exists a weak solution to (2.2).*

Proof. Recall the function β introduced in Section 3.2. Fix $x \in \mathbb{R}^d$. In order to prove the existence of a weak solution of (2.2) it suffices to construct continuous stochastic processes X, W, L taking values in $\mathbb{R}^d, \mathbb{R}^d$ and \mathbb{R} respectively on some probability space such that

- W is a standard d -dimensional Brownian motion adapted to the filtration

$$\{\mathcal{F}_t = \sigma\{X(\cdot \wedge t), W(\cdot \wedge t), L(\cdot \wedge t)\}\}_{t \geq 0}$$

and having increments $W_{t_1} - W_t$ independent of $\mathcal{F}_t, t_1 > t$.

- For all $t \geq 0$

$$X(t) = x + W(t) + \int_0^t \beta(Y(s)) dL(s)$$

where $X(t) = (U(t), Y(t)) \in \mathbb{R} \times \mathbb{R}^{d-1}$,

- L is the symmetric local time of U at 0.

Write $\beta(\xi) = (\beta_1(\xi), \dots, \beta_d(\xi))$, $\xi \in \mathbb{R}^{d-1}$. For $\xi \in \mathbb{R}^{d-1}$ and $i = 2, \dots, d$, let $\bar{\beta}_i(\xi) = 2\lfloor(\beta_i(\xi) + 1)/2\rfloor$ and $\hat{\beta}_i(\xi) = \beta_i(\xi) - \bar{\beta}_i(\xi)$. Note that $\bar{\beta}_i$ takes values in $2\mathbb{Z}$ and $\hat{\beta}_i$ in $[-1, 1]$. Let

$$p_{i,1}(\xi) = \frac{1 + \hat{\beta}_i(\xi)}{2}, \quad p_{i,-1}(\xi) = \frac{1 - \hat{\beta}_i(\xi)}{2}, \quad i = 2, \dots, d, \xi \in \mathbb{R}^{d-1}.$$

Also let

$$p_{1,1}(\xi) = \frac{1 + \beta_1(\xi)}{2}, \quad p_{1,-1}(\xi) = \frac{1 - \beta_1(\xi)}{2}, \quad \xi \in \mathbb{R}^{d-1}.$$

Let $\{x^n\}_{n \in \mathbb{N}}$ be a sequence in \mathbb{Z}^d such that $\bar{x}^n = x^n/\sqrt{n}$ converges to x . For each n define a Markov chain $\{X_k^n\}_{k \in \mathbb{N}_0}$ such that $X_0^n = x^n$ and for $k \geq 0, v \in \mathbb{Z}^d$

$$\begin{aligned} P(\Delta X_k^n = u \mid X_k^n = v) &= \frac{1}{2^d}, \quad u \in \{-1, 1\}^d, \text{ if } v_1 \neq 0 \\ P\left(\Delta X_k^n = \sum_{i=2}^d e_i \bar{\beta}_i(v'/\sqrt{n}) + u \mid X_k^n = v\right) \\ &= \prod_{i=1}^d p_{i,u_i}(v'/\sqrt{n}), \quad u = (u_1, \dots, u_d) \in \{-1, 1\}^d, \text{ if } v_1 = 0. \end{aligned} \tag{3.6}$$

Here $v = (v_1, v') \in \mathbb{Z} \times \mathbb{Z}^{d-1}$, $\{e_i\}_{i=1}^d$ is the standard coordinate basis in \mathbb{R}^d and $\Delta Z_k = Z_{k+1} - Z_k$ for an \mathbb{R}^d -valued sequence $\{Z_k\}$. Since the transition probabilities of this chain agree with those of a homogeneous random walk (RW) away from $\{v_1 = 0\}$ we can couple this chain with a d -dimensional RW $\{W_k^n\}_{k \in \mathbb{N}_0}$ such that the following identity holds:

$$X_k^n = x^n + \sum_{i=0}^{k-1} \left(\Delta W_i^n 1_{U_i^n \neq 0} + \Delta X_i^n 1_{U_i^n = 0} \right) \tag{3.7}$$

where we write $X_i^n = (U_i^n, Y_i^n) \in \mathbb{R} \times \mathbb{R}^{d-1}$. Letting $\mathcal{F}_k^n = \sigma\{X_j^n, W_j^n, j \leq k\}$ we have

$$E(\Delta X_k^n \mid \mathcal{F}_k^n) = \beta(Y_k^n/\sqrt{n}) 1_{U_k^n = 0}.$$

We can therefore write

$$X_k^n = x^n + W_k^n + \sum_{i=0}^{k-1} (\beta(Y_i^n/\sqrt{n}) + M_i^n) 1_{U_i^n = 0} \tag{3.8}$$

where $|M_i^n| \leq (3 + 2\sup_{\xi} |\beta(\xi)|)\sqrt{d} \equiv c_1$ and $\hat{M}_k^n = \sum_{i=0}^{k-1} M_i^n 1_{U_i^n = 0}$ is a martingale.

Define $L_0^n = 0$ and for $k \geq 1, L_k^n = \sum_{i=0}^{k-1} 1_{U_i^n = 0}$. Note that the increasing process associated with \hat{M}^n is bounded by $2c_1^2 L^n$, and thus by Burkholder’s inequality,

$$E[\max_{i \leq k} (\hat{M}_i^n)^2] \leq c_2 E[L_k^n] \tag{3.9}$$

where $c_2 = 8c_1^2$. Next note that

$$\Delta|U_k^n| = \Delta U_k^n 1_{U_k^n > 0} - \Delta U_k^n 1_{U_k^n < 0} + 1_{U_k^n = 0}.$$

Thus

$$|U_k^n| = |x_1^n| + Z_k^n + L_k^n, \tag{3.10}$$

where $Z_k^n = \sum_{i=0}^{k-1} \text{sgn}(U_i^n) \Delta U_i^n$. Recalling (3.6) and that $\text{sgn}(0) = 0$ we see that

$$E(Z_k^n) = E\left(\sum_{i=0}^{k-1} 1_{U_i^n \neq 0} \text{sgn}(U_i^n) E(\Delta U_i^n \mid \mathcal{F}_i^n)\right) = 0. \tag{3.11}$$

Next let $\{\zeta_i\}$ be an i.i.d. sequence of $\{-1, 1\}$ -valued random variables, independent of all the sequences introduced above, such that $P(\zeta_i = 1) = P(\zeta_i = -1) = 1/2$ and let

$$Z_k^{*,n} = Z_k^n + \sum_{i=0}^{k-1} \zeta_i 1_{U_i^n=0}.$$

By construction $\{Z_k^{*,n}\}$ is a simple random walk on \mathbb{Z} . Also, as for (3.9)

$$E[\max_{i \leq k} (Z_i^{*,n} - Z_i^n)^2] \leq 8E[L_k^n]. \tag{3.12}$$

Define now a continuous time process

$$\bar{W}^n(t) = \frac{1}{\sqrt{n}} W_{\lfloor nt \rfloor}^n, \quad t \in [0, \infty)$$

and use the same rescaling to define continuous time versions $\bar{X}^n, \bar{U}^n, \bar{Y}^n, \bar{L}^n, \bar{Z}^n, \bar{Z}^{*,n}$ of the discrete time processes $X^n, U^n, Y^n, L^n, Z^n, Z^{*,n}$. Then, from (3.8),

$$\bar{X}^n(t) = \bar{x}^n + \bar{W}^n(t) + \int_{[0, \frac{\lfloor nt \rfloor}{n}]} \beta(\bar{Y}^n(s-)) d\bar{L}^n(s) + \varepsilon^n(t), \tag{3.13}$$

where by (3.9)

$$E[\max_{s \leq t} (\varepsilon^n(s))^2] \leq \frac{c_2}{\sqrt{n}} E[\bar{L}^n(t)]. \tag{3.14}$$

By (3.10) and (3.11) we have

$$E(\bar{L}^n(t)) \leq E|U^n(t)| \leq |\bar{x}_1^n| + \sqrt{t},$$

where the last inequality follows on observing that $|U^n|$ has the same law as the absolute value of a RW starting from \bar{x}_1^n . Thus

$$\sup_n E(\bar{L}^n(t)) < \infty \quad \text{for all } t \geq 0. \tag{3.15}$$

Using this in (3.14) we get that ε^n converges to 0 in probability, uniformly on compacts (u.o.c.). Also, recalling that $\{Z_k^{*,n}\}$ is a simple random walk, we have by Donsker’s theorem that $\bar{Z}^{*,n}$ converges in distribution in $\mathbb{D}_{\mathbb{R}}$ to a standard Brownian motion Z . From (3.12) and (3.15) we have that $\bar{Z}^{*,n} - \bar{Z}^n$ converges to 0 in probability, u.o.c. and so \bar{Z}^n converges to Z weakly in $\mathbb{D}_{\mathbb{R}}$ as well.

Consider the 1-dimensional Skorohod map $\Gamma : \mathbb{D}_{\mathbb{R}} \rightarrow \mathbb{D}_{\mathbb{R}_+}$,

$$\Gamma(g)(t) = g(t) - \inf_{0 \leq s \leq t} (g(s) \wedge 0), \quad t \geq 0, \quad g \in \mathbb{D}_{\mathbb{R}}$$

and recall that $-\inf_{0 \leq s \leq t} (g(s) \wedge 0)$ is the unique nonnegative, nondecreasing RCLL function h that satisfies $f(t) = g(t) + h(t) \geq 0$ for all $t \geq 0$ and $\int_{[0, \infty)} 1_{(0, \infty)}(f(t)) dh(t) = 0$. Writing

$$|\bar{U}^n(t)| = \left(|\bar{x}_1^n| + \bar{Z}^n(t) - \frac{1}{\sqrt{n}} 1_{\bar{U}^n(t)=0} \right) + \bar{L}^n \left(\frac{\lfloor nt \rfloor + 1}{n} \right),$$

we see that

$$|\bar{U}^n(t)| = \Gamma(|\bar{x}_1^n| + \bar{Z}^n)(t), \quad \text{where } \tilde{Z}^n(t) = \bar{Z}^n(t) - \frac{1}{\sqrt{n}} 1_{\bar{U}^n(t)=0}.$$

Since $|\tilde{Z}^n(t) - \bar{Z}^n(t)| \leq 1/\sqrt{n}$, we have that \tilde{Z}^n converges to Z weakly in $\mathbb{D}_{\mathbb{R}}$. By the continuity of Γ , we now have that $(\tilde{Z}^n, |\bar{U}^n|, \bar{L}^n)$ converges in distribution in $\mathbb{D}_{\mathbb{R}^3}$ to (Z, R, L) , where

$$R = |x_1| + Z - \min_{s \leq \cdot} [(|x_1| + Z(s)) \wedge 0], \quad L = -\min_{s \leq \cdot} [(|x_1| + Z(s)) \wedge 0].$$

Next, using again Donsker’s theorem, \bar{W}^n converges in distribution to a d -dimensional standard Brownian motion. In particular, \bar{W}^n is C -tight. Combining this with (3.13) and the weak convergence of ε^n to 0 and \bar{L}^n to L we see that \bar{X}^n is C -tight as well. Denote by (W, X, U, Y, Z, R, L) a subsequential weak limit of $(\bar{W}^n, \bar{X}^n, \bar{U}^n, \bar{Y}^n, \bar{Z}^n, |\bar{U}^n|, \bar{L}^n)$. The integral in (3.13) converges to $\int_0^\cdot \beta(Y(s)) dL(s)$ by Theorem 2.2 of [7]. Indeed, from (3.15) and since β is bounded

$$\sup_{0 \leq t \leq T} \left| \int_{[0, \frac{\lfloor nt \rfloor]}{n}]} \beta(\bar{Y}^n(s-)) d\bar{L}^n(s) - \int_{[0, t]} \beta(\bar{Y}^n(s-)) d\bar{L}^n(s) \right| \rightarrow 0$$

in probability as $n \rightarrow \infty$, for every $T > 0$. We now note that from (3.15), Condition C2.2(i) in Theorem 2.2 of [7] is satisfied on taking $X_n, Y_n, \delta, M_n^\delta, A_n^\delta, \tau_n^\alpha$ there equal to $\beta(\bar{Y}^n), \bar{L}^n, \infty, 0, \bar{L}^n, \infty$, respectively, for all $\delta, \alpha > 0$.

In fact we have the joint convergence of

$$\left(\bar{W}^n, \bar{X}^n, \int_{[0, \frac{\lfloor nt \rfloor]}{n}]} \beta(\bar{Y}^n(s-)) d\bar{L}^n(s) \right)$$

to $(W, X, \int_0^\cdot \beta(Y(s)) dL(s))$. We therefore obtain

$$X(t) = x + W(t) + \int_0^t \beta(Y(s)) dL(s), \quad t \geq 0.$$

Since R is the limit of $|\bar{U}^n|$, the relation $R = |U|$ must hold. Also $X = (U, Y)$. For a continuous and bounded function $G : \bar{C} \times \mathbb{C}_{\mathbb{R}} \rightarrow \mathbb{R}$ and $0 \leq t \leq s < \infty$

$$|E(G(\bar{W}^n(\cdot \wedge t), \bar{X}^n(\cdot \wedge t), \bar{L}^n(\cdot \wedge t), \bar{Z}^n(\cdot \wedge t)) [\bar{W}^n(s) - \bar{W}^n(t)])| \leq \frac{2\|G\|_\infty}{\sqrt{n}}$$

and thus

$$E(G(\bar{W}(\cdot \wedge t), \bar{X}(\cdot \wedge t), \bar{L}(\cdot \wedge t), \bar{Z}(\cdot \wedge t)) [\bar{W}(s) - \bar{W}(t)]) = 0.$$

Consequently W is a standard d -dimensional \mathcal{F}_t -Brownian motion where $\{\mathcal{F}_t = \sigma\{X(\cdot \wedge t), W(\cdot \wedge t), L(\cdot \wedge t), Z(\cdot \wedge t)\}\}_{t \geq 0}$. Similarly Z is a 1-dimensional \mathcal{F}_t -Brownian motion. Next note that

$$|U(t)| = |x_1| + Z(t) + L(t)$$

and by Tanaka’s formula, since L increases only when $|U| = 0$,

$$|U(t)| = |x_1| + \int_0^t \operatorname{sgn}(U(s))dW_1(s) + L^U(t),$$

where L^U is the symmetric local time of U at 0. Combining the above two displays we have $L = L^U$. Thus the processes (X, W, L) satisfy all the properties listed at the beginning of the subsection and consequently we have proved the existence of a weak solution. \square

4. Brownian particles with skew-elastic collisions

As an application we consider a model introduced by Fernholz et al. [4] for the dynamics of a pair of 1-dimensional Brownian particles X_1 and X_2 that exhibit various possible types of interaction when they collide. The equations involve the local time at zero of the relative position, and the types of interaction are determined by the coefficients in front of the local time terms. For a continuous real semimartingale Z , let L_+^Z be defined by the relation

$$L_+^Z(t) = Z^+(t) - Z^+(0) - \int_0^t 1_{\{Z(s)>0\}}dZ(s)$$

and let $L_-^Z = L_+^{-Z}$. Then the symmetric local time of Z at 0, L^Z , is given as $L^Z = L_+^Z + L_-^Z$. The motion of the particles is described by the set of equations

$$\begin{aligned} dX_1(t) &= k_1(X(t))dt + \frac{1}{\sqrt{2}}dB_1(t) + \frac{1 - \zeta_1(X(t))}{2}dL_+^{X_1 - X_2}(t) \\ &\quad + \frac{1 - \eta_1(X(t))}{2}dL_+^{X_2 - X_1}(t) \\ dX_2(t) &= k_2(X(t))dt + \frac{1}{\sqrt{2}}dB_2(t) + \frac{1 - \zeta_2(X(t))}{2}dL_+^{X_1 - X_2}(t) \\ &\quad + \frac{1 - \eta_2(X(t))}{2}dL_+^{X_2 - X_1}(t). \end{aligned} \tag{4.1}$$

Here (B_1, B_2) is a planar Brownian motion, $X = (X_1, X_2)$ and $k_i, \zeta_i, \eta_i, i = 1, 2$, are bounded measurable functions from \mathbb{R}^2 to \mathbb{R} (additional conditions on $\zeta_i, \eta_i, i = 1, 2$ will be introduced below). In [4] the coefficients ζ_i and η_i are assumed to be constant rather than state-dependent. Different types of behavior are obtained by different choices of these constants. For example, when $\zeta_1 = \eta_1 = 1$ (resp., $\zeta_2 = \eta_2 = 1$), the local times disappear from the equation for X_1 (resp., X_2) and so the collisions do not affect the first (resp., second) particle. When both conditions hold, the motions are completely frictionless. On the other extreme, when $\zeta_2 = \eta_2 = 1$ and $\zeta_1 = \eta_1 = -1$, the trajectory of X_1 bounces off that of X_2 as if it were a perfectly reflecting boundary. See [4] for more general conditions under which frictionless motion and perfect reflection are attained. Other combinations of the constants give rise to a whole range of elastic collisions.

The equations presented here have state-dependent local time coefficients. This allows one to model variability in the type of collision, where the type is determined by the collision position. (The paper [4] also emphasizes rank-dependent motion, and to this end the drift and diffusion coefficients considered are constant on each of the sets $\{X_1 \leq X_2\}$ and $\{X_1 > X_2\}$. Such a drift can be realized in our model by selecting k_i suitably. However, we only consider constant diffusion coefficient here.) For the equations with constant local time coefficients, [4] obtain strong existence and pathwise uniqueness. Our goal here is to exhibit weak existence and uniqueness for Eqs. (4.1).

For $x \in \mathbb{R}^2$ let

$$\begin{aligned} \zeta(x) &= 1 + \frac{\zeta_1(x) - \zeta_2(x)}{2}, & \eta(x) &= 1 - \frac{\eta_1(x) - \eta_2(x)}{2}, \\ \bar{\zeta}(x) &= 1 - \frac{\zeta_1(x) + \zeta_2(x)}{2}, & \bar{\eta}(x) &= 1 - \frac{\eta_1(x) + \eta_2(x)}{2}. \end{aligned}$$

We assume that for all $x \in \mathbb{R}^2$,

$$\zeta(x) \geq 0, \quad \eta(x) \geq 0, \quad \eta(x) + \zeta(x) \neq 0, \quad \alpha(x) = \frac{\eta(x)}{\eta(x) + \zeta(x)} \in [0, 1].$$

Consider the equations associated with the linear transformations

$$Y = X_1 - X_2, \quad U = X_1 + X_2.$$

If $X = (X_1, X_2)$ solves (4.1) then (Y, U) will solve

$$\begin{aligned} dY(t) &= (k_1(X(t)) - k_2(X(t)))dt + dW_1(t) + (1 - \zeta(X(t)))dL_+^Y(t) \\ &\quad + (\eta(X(t)) - 1)dL_-^Y(t) \\ dU(t) &= (k_1(X(t)) + k_2(X(t)))dt + dW_2(t) + \bar{\zeta}(X(t))dL_+^Y(t) + \bar{\eta}(X(t))dL_-^Y(t), \end{aligned} \tag{4.2}$$

where $W_1 = \frac{1}{\sqrt{2}}(B_1 - B_2)$, $W_2 = \frac{1}{\sqrt{2}}(B_1 + B_2)$. Using the relations

$$L_+^Y(t) - L_-^Y(t) = \int_0^t 1_{\{Y(s)=0\}}dY(s), \quad L^Y(t) = (L_+^Y(t) + L_-^Y(t)), \quad t \geq 0,$$

and $\int_{[0,\infty)} 1_{\{Y(s)=0\}}ds = 0$, we see that

$$L_+^Y(t) = \int_0^t \alpha(X(s))dL^Y(s), \quad L_-^Y(t) = \int_0^t (1 - \alpha(X(s)))dL^Y(s).$$

Define $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as $\psi(u, y) = (\frac{u+y}{2}, \frac{u-y}{2})$. Then (4.2) can be rewritten as

$$\begin{aligned} dY(t) &= a_1(U(t), Y(t))dt + dW_1(t) + b_1(U(t))dL^Y(t) \\ dU(t) &= a_2(U(t), Y(t))dt + dW_2(t) + b_2(U(t))dL^Y(t), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} a_1(u, y) &= (k_1 - k_2) \circ \psi(u, y), & a_2(u, y) &= (k_1 + k_2) \circ \psi(u, y), \\ b_1(u) &= \beta_1 \circ \psi(u, 0), & b_2(u) &= \beta_2 \circ \psi(u, 0), \\ \beta_1(x) &= (2\alpha(x) - 1), & \beta_2(x) &= \bar{\zeta}(x)\alpha(x) + \bar{\eta}(x)(1 - \alpha(x)). \end{aligned}$$

The following is now an immediate consequence of Theorem 2.1.

Theorem 4.1. *Suppose that $\zeta_i, \eta_i, i = 1, 2$ and α are Lipschitz. Then Eqs. (4.3) and (4.1), with any given initial conditions, have a unique weak solution.*

Proof. Note our assumptions ensure that b is Lipschitz with b_1 taking values in $[-1, 1]$. Thus existence of a unique weak solution of (4.3) is an immediate consequence of Theorem 2.1. Unique solvability of (4.1) follows on observing that (X_1, X_2) solves (4.1) with driving Brownian motions (B_1, B_2) if and only if (Y, U) solves (4.3) with driving Brownian motions (W_1, W_2) , where $Y = X_1 - X_2, U = Y_1 + Y_2, W_1 = \frac{1}{\sqrt{2}}(B_1 - B_2), W_2 = \frac{1}{\sqrt{2}}(B_1 + B_2)$. \square

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