On near optimal trajectories for a game associated with the ∞-Laplacian

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Abstract: A two-player stochastic differential game representation has recently been obtained for solutions of the equation $-\Delta_\infty u = h$ in a $\mathcal{C}^2$ domain with Dirichlet boundary condition, where h is continuous and takes values in $\mathbb{R} \setminus \{0\}$. Under appropriate assumptions, including smoothness of u, we identify a family of diffusion processes that may arise as the vanishing δ limit law of the state process, when both players play δ-optimally. We also identify the limit law of the state process under a sequence of near saddle points.

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1. Introduction and main result

Consider the equation

$$\begin{cases} -2\Delta_\infty u = h & \text{in } G, \\[2mm] u = g & \text{on } \partial G, \end{cases} \tag{1.1}$$

where, for an integer $m \geq 2$, $G \subset \mathbb{R}^m$ is a bounded $\mathcal{C}^2$ domain, and $g \in \mathcal{C}(\partial G, \mathbb{R})$ and the functions $h \in \mathcal{C}(\bar{G}, \mathbb{R} \setminus \{0\})$ are given. The infinity-Laplacian is defined as

$$\Delta_\infty f = \frac{1}{|Df|^2} \sum_{i,j=1}^m D_i f \, D_{ij} f \, D_j f = \frac{Df'}{|Df|} \, D^2 f \, \frac{Df}{|Df|},$$

provided $Df \neq 0$, where for a $\mathcal{C}^2$ function f we denote by Df the gradient and by $D^2 f$ the Hessian matrix. We refer the

reader to [1, 2, 4, 6, 7, 10] for background on the infinity-Laplacian and some related PDE theory. This paper is motivated by recent work of Peres et. al. [8], where a discrete time

random turn game, referred to as *Tug-of-War*, is developed in relation to (1.1). This game, parameterized by $\varepsilon > 0$, has the property that the vanishing-ε limit of the value function uniquely solves (1.1) in the viscosity sense (a result that is valid also in the homogenous case, $h = 0$, excluded from the current paper). The stochastic differential equation (SDE)

$$dX_t = 2\bar{p}(X_t)dW_t + 2q(X_t)dt, \qquad (1.2)$$

where

$$\bar{p} = \frac{Du}{|Du|}, \qquad q = \frac{1}{|Du|^2}(D^2u\,Du - \Delta_\infty u\,Du), \qquad (1.3)$$

is suggested in [8] as the game's dynamics in the vanishing-ε limit. The relation is rigorously established in examples, but only heuristically justified in general. Denote $p = Du$. Note that $p \cdot q = 0$ in G, and thus (1.2) is a special case of the family of SDE

$$dX_t = 2\bar{p}(X_t)dW_t + 2r(X_t)dt, \qquad (1.4)$$

as r ranges over R, where

$$R = \{r \in \mathcal{C}(\bar{G} : \mathbb{R}^m) : r \cdot p = 0\}.$$

In [3], a two-player zero-sum stochastic differential game (SDG) is considered, for which the value function uniquely solves (1.1) in the viscosity sense. The goal of the present paper is to show that, under appropriate conditions, (1.2) can be rigorously interpreted as the optimal dynamics of the SDG. In fact we will establish a stronger result, namely, that *any* member of the family of SDE (1.4) (satisfying some regularity conditions) can be realized as the optimal dynamics of the SDG. Defined in the Elliott-Kalton sense, the SDG of [3] is formulated in such a way that one of the players selects a strategy, and then the other selects a control process (see Definition 1.1 below). We will assume in this paper that the equation possesses a classical solution u i.e., $\mathcal{C}^2$ with non-vanishing gradient. Given any $r \in R$, we specify, for each $\delta > 0$, a δ-optimal strategy β^δ, and a control process Y^δ that is δ-optimal for play against β^δ (such a (β^δ, Y^δ) is referred to as a δ-optimal play), in terms of r and the first and second derivatives of u. We then identify the vanishing δ limit law of the state process under (β^δ, Y^δ), as the solution X to the SDE (1.4), stopped when X hits the boundary ∂G.

A stronger result, of identifying the limit under any vanishing δ sequence of δ-optimal plays, is of interest but appears to be difficult, and is not treated in this paper.

The construction of near optimal strategy-control pairs is based on an interpretation of (1.1) as the following Bellman-Isaacs type equation (see also (2.4))

$$\sup_{|b|=1,d\geq 0} \inf_{|a|=1,c\geq 0} \left\{ -\frac{1}{2}(a-b)'(D^2u)(a-b) - (c+d)(a+b) \cdot Du \right\} = h. \qquad (1.5)$$

In this form there is a natural way to construct strategy and control, by associating the supremum and infimum with the two players. The variables a, b, c and d selected by the players dictate the coefficients of the game's state process, and, as we prove, the coefficients converge to those of equation (1.4) in the limit as the supremum and infimum are achieved. This convergence is then lifted to the convergence of the underlying processes to the diffusion (1.4).

Although the SDE (1.4) represents an optimal play for every $r \in R$, the case $r = q$ (1.2) does have a special role in our proof. The supremum in (1.5) can be reached within precision δ by selecting $b = -Du/\|Du\|$ and a large constant $d = d^\delta$. For this choice of (b, d), the infimum in (1.5) can be *attained* by selecting $c = 0$ and some particular a, which we will denote a^δ (see Proposition 2.1 as well as the construction of the map a^δ in (2.22)). The case of the SDE with $r = q$ arises as the limiting trajectory when the minimizing player selects a *nearly optimal* strategy based on the selection (b, d^δ), and the maximizing player responds by choosing a control process according to the *optimal* map $(a^\delta, 0)$. However, it turns out that the notion of δ-optimality leaves enough room for other limit processes to arise. Indeed, a general r is achieved upon introducing a small perturbation of the map a^δ to the action of the minimizing player.

The plays (β^δ, Y^δ) described above do not represent saddle points in the sense of game theory. Indeed Y^δ is only guaranteed to be δ-optimal when played against β^δ. To analyze approximate saddle points for the game, we consider a game formulation in which the two players choose controls Y and Z that are only allowed to depend on the game's history via 'feedback' from the state through suitable maps $\mathbf{y}, \mathbf{z}$, mapping the state space to the control space. For this formulation of the game, we show that the case $r = 0$ is obtained as the limit state process under a sequence of plays in which $(\mathbf{y}, \mathbf{z})$ represents a near saddle point.

Solutions of (1.1), in general, need not be $\mathcal{C}^2$. The study of regularity properties of solutions of ∞-Laplace equations is a research program in progress. The papers [5, 9] have recently established $\mathcal{C}^{1,\alpha}$ smoothness of solutions in planar domains and outlined some general approaches for treating the problem in dimension 3 and higher. It is of interest to identify near optimal plays without making smoothness assumptions on u. In particular, since only the first derivative of u appears in the coefficients of (1.4), it is natural to conjecture that Theorem 1.1 holds for $\mathcal{C}^1$ solutions (in the viscosity sense). Our proof, however, relies on $\mathcal{C}^2$ smoothness. It is tempting to approach the problem for non-$\mathcal{C}^2$ solutions by using the recipe for constructing δ-optimal pairs, obtained in this work, on $\mathcal{C}^2$ approximations to the solution. As we discuss at the end of this section, the success of such an approach appears to depend upon an open problem (Problem 1.1 below).

In the rest of this section, we describe the setting and state the main results. Throughout, we will make the following

Assumption 1.1. *There exists a $\mathcal{C}^2(\bar{G})$ function u, with $Du \neq 0$ on $\bar{G}$, that solves (1.1) in the classical sense.*

As a consequence of [8], that proves uniqueness (and existence) of viscosity solutions to (1.1) under the above assumptions on h, the function u of Assumption 1.1 is the unique classical solution of (1.1).

We now present the SDG. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \boldsymbol{P})$ be a complete filtered probability space with right-continuous filtration, supporting an $(m + 1)$-dimensional $\{\mathcal{F}_t\}$-Brownian motion $\overline{W} = (W, \widetilde{W})$, where W and $\widetilde{W}$ are 1- and m-dimensional Brownian motions, respectively. Denote

by $\boldsymbol{E}$ the expectation with respect to $\boldsymbol{P}$. Let X_t be a process taking values in $\mathbb{R}^m$, given by

$$X_t = x + \int_0^t (A_s - B_s)dW_s + \int_0^t (C_s + D_s)(A_s + B_s)ds, \qquad t \in [0, \infty), \qquad (1.6)$$

where $x \in \overline{G}$, A_t and B_t take values in the unit sphere $\mathcal{S}^{m-1} \subset \mathbb{R}^m$, and C_t and D_t take values in $[0, \infty)$. Denote

$$Y^0 = (A, C), \quad Z^0 = (B, D). \qquad (1.7)$$

The processes Y^0 and Z^0 take values in $\mathcal{H} = \mathcal{S}^{m-1} \times [0, \infty)$. These processes will correspond to control actions of the maximizing and minimizing player, respectively. For a process $H^0 = (A, C)$ taking values in $\mathcal{H}$ we let $S(H^0) = \operatorname{ess\,sup} \sup_{t \in [0, \infty)} C_t$. In the formulation below, each player initially declares a bound S, and then plays so as to keep $S(H^0) \leq S$.

Definition 1.1. *(i) A pair $H = (\{H_t^0\}, S)$, where $S \in \mathbb{N}$ and $\{H_t^0\}$ is a process taking values in $\mathcal{H}$, is said to be an admissible control if $\{H_t^0\}$ is $\{\mathcal{F}_t\}$-progressively measurable, and $S(H^0) \leq S$. The set of all admissible controls is denoted by M. For $H = (\{H_t^0\}, S) \in M$, denote $\boldsymbol{S}(H) = S$. (ii) A mapping $\varrho : M \to M$ is said to be a strategy if, for every t, and $H, \widetilde{H} \in M$,*

$$\boldsymbol{P}(H_s^0 = \widetilde{H}_s^0 \text{ for a.e. } s \in [0,t]) = 1 \text{ and } S = \widetilde{S}$$

implies

$$\boldsymbol{P}(I_s^0 = \widetilde{I}_s^0 \text{ for a.e. } s \in [0,t]) = 1 \text{ and } T = \widetilde{T},$$

where $(I^0, T) = \varrho[(H^0, S)]$ and $(\widetilde{I}^0, \widetilde{T}) = \varrho[(\widetilde{H}^0, \widetilde{S})]$. The set of all strategies is denoted by $\widetilde{\Gamma}$. For $\varrho \in \widetilde{\Gamma}$, let $\boldsymbol{S}(\varrho) = \sup_{H \in M} \boldsymbol{S}(\varrho[H])$. Let

$$\Gamma = \{\varrho \in \widetilde{\Gamma} : \boldsymbol{S}(\varrho) < \infty\}.$$

Note that the Brownian motion $\widetilde{W}$ does not appear explicitly in the state dynamics, however the control processes may depend on $\widetilde{W}$. Such a formulation where the underlying filtration is rich enough to support an $(m+1)$-dimensional Brownian motion originates from Swiech's construction [12], and is crucially used in the proof of wellposedness of the SDG (see [3] for details).

We will use the symbols Y and α for a generic control and strategy for the maximizing player, and Z and β will denote the same for the minimizing player. For a process ξ taking values in $\mathbb{R}^m$ and a set $A \subset \mathbb{R}^m$, we will write $\tau_A(\xi)$ for

$$\inf\{t \geq 0 : \xi_t \notin A\}$$

(where the infimum over an empty set is ∞). Let

$$\tau = \tau_G(X).$$

We write

$$X(x, Y^0, Z^0) \qquad [\text{resp.}, \tau(x, Y^0, Z^0)] \qquad (1.8)$$

for the process X [resp., the random time τ] when it is important to specify the explicit dependence on (x, Y^0, Z^0). If $\tau < \infty$ a.s., then the payoff $J(x, Y^0, Z^0)$ is well defined with values in $[-\infty, \infty]$, where

$$J(x, Y^0, Z^0) = \boldsymbol{E}\left[\int_0^\tau h(X_s)ds + g(X_\tau)\right], \tag{1.9}$$

and X is given by (1.6). When $\boldsymbol{P}(\tau(x, Y^0, Z^0) = \infty) > 0$, we set, consistent with the expectation of the first term in (1.9), $J(x, Y^0, Z^0)$ to be $+\infty$ $[-\infty]$ for the case $h > 0$ [resp., $h < 0$].

If $Y = (Y^0, K), Z = (Z^0, L) \in M$, we sometimes write $J(x, Y, Z) = J(x, (Y^0, K), (Z^0, L))$ for $J(x, Y^0, Z^0)$. Similar conventions will be used for $X(x, Y, Z)$ and $\tau(x, Y, Z)$. Occasionally, with an abuse of terminology, when $Y = (Y^0, K) \in M$, we will write $Y^0 \in M$. Let

$$J^x(Y, \beta) = J(x, Y, \beta[Y]), \qquad x \in \bar{G}, \ Y \in M, \ \beta \in \Gamma,$$

$$J^x(\alpha, Z) = J(x, \alpha[Z], Z), \qquad x \in \bar{G}, \ \alpha \in \Gamma, \ Z \in M.$$

Define analogously $X^x(Y, \beta)$, $X^x(\alpha, Z)$, $\tau^x(Y, \beta)$ and $\tau^x(\alpha, Z)$ via (1.8). Define the lower value of the SDG by

$$V(x) = \inf_{\beta \in \Gamma} \sup_{Y \in M} J^x(Y, \beta), \tag{1.10}$$

and the upper value by

$$U(x) = \sup_{\alpha \in \Gamma} \inf_{Z \in M} J^x(\alpha, Z). \tag{1.11}$$

The game is said to have value if $U = V$.

Theorem 1.1 of [3] shows that the SDG has value, and that $U = V = u$ on $\bar{G}$.

Let $x \in \bar{G}$ and $\delta > 0$ be given. We say that a policy $\beta \in \Gamma$ is δ-optimal for the lower game and initial condition x if $\sup_{Y \in M} J^x(Y, \beta) \leq V(x) + \delta$. When a strategy $\beta \in \Gamma$ is given, we say that a control $Y \in M$ is δ-optimal for play against β with initial condition x, if $J^x(Y, \beta) \geq \sup_{Y' \in M} J^x(Y', \beta) - \delta$. A pair (Y, β) is said to be a δ-optimal play for the lower game with initial condition x, if β is δ-optimal for the lower game and Y is δ-optimal for play against β (both considered with initial condition x). Note that for such a (Y, β) pair

$$J^x(Y, \beta) - \delta \leq V(x) \leq J^x(Y, \beta) + \delta.$$

An (α, Z) δ-optimal play for the upper game with initial condition x is defined in a similar manner.

Our first main result is the following.

Theorem 1.1. *Let Assumption 1.1 hold. Fix $r \in R$. Assume, in addition, that there exist uniformly continuous, bounded extensions of $\bar{p}$ and r to all of $\mathbb{R}^m$ such that, for every $x \in \mathbb{R}^m$, weak uniqueness holds for solutions of (1.4) starting from x. Fix $x \in \bar{G}$ and let X and τ denote such a solution and, respectively, the corresponding exit time from G. Then, given any sequence $\{\delta_n\}_{n \geq 1}$, $\delta_n \downarrow 0$, there exists a sequence of strategy-control pairs $(\beta^n, Y^n) \in \Gamma \times M$, $n \geq 1$, with the following properties.*

     i. For every n, the pair (β^n, Y^n) forms a δ_n-optimal play for the lower game with initial condition x;

     ii. Denoting $X^n = X^x(Y^n, \beta^n)$ and $\tau^n = \tau_G(X^n)$, one has that $(X^n(\cdot \wedge \tau^n), \tau^n)$ converges in distribution to $(X(\cdot \wedge \tau), \tau)$, as a sequence of random variables with values in $C([0, \infty) : \bar{G}) \times [0, \infty]$.

An analogous result holds for the upper game.

Remark 1.1. One can always find uniformly continuous bounded extensions of $\bar{p}$ and r, however, in general, without additional conditions weak uniqueness may not hold. A sufficient condition for uniqueness to hold is that p and r are Lipschitz on $\bar{G}$, since then these functions admit a Lipschitz extension to $\mathbb{R}^m$. In the case $r = q$, a sufficient condition is that $D^2 u$ is Lipschitz on $\bar{G}$, since then $\bar{p}$ and q are Lipschitz.

Next we introduce the notion of an approximate saddle point. There appears to be no natural definition of a saddle point in a formulation where each player selects two different types of actions (a strategy and a control) in the two games (upper and lower). In this regard, a formulation based on 'feedback' controls, that we now describe, is more convenient.

Denote by $\mathbf{H}$ the set of all pairs $\mathbf{y} = (a, c)$, where a and c are Lipschitz maps from $\bar{G}$ to $\mathcal{S}^{m-1}$ and $(0, \infty)$, respectively. Given $x \in \bar{G}$ and a pair $(\mathbf{y}, \mathbf{z}) \in \mathbf{H}^2$, there exists a unique process $X = X(x, \mathbf{y}, \mathbf{z})$ such that

$$X(t) = x + \int_0^t (a(X_s) - b(X_s))dW_s + \int_0^t (c(X_s) + d(X_s))(a(X_s) + b(X_s))ds, \qquad t \geq 0, \, a.s.$$

Let $J(x, \mathbf{y}, \mathbf{z})$ represent the corresponding cost (1.9), with the same convention regarding the case $\mathbf{P}(\tau = \infty) > 0$ as before. One can then talk of a game where the two players select the feedback functions $\mathbf{y}$ and $\mathbf{z}$, and define the corresponding lower value

$$V^{\mathrm{fb}}(x) = \sup_{\mathbf{y} \in \mathbf{H}} \inf_{\mathbf{z} \in \mathbf{H}} J(x, y, z),$$

and upper value

$$U^{\mathrm{fb}}(x) = \inf_{\mathbf{z} \in \mathbf{H}} \sup_{\mathbf{y} \in \mathbf{H}} J(x, y, z).$$

Note that $V^{\mathrm{fb}}(x) \leq U^{\mathrm{fb}}(x)$. When the two are equal, we can find, for every $\delta > 0$, a $\mathbf{y}^*$ [$\mathbf{z}^*$] that is δ-optimal for the lower [resp., upper] game, so that

$$J(x, \mathbf{y}, \mathbf{z}^*) - \delta \leq V^{\mathrm{fb}}(x) = U^{\mathrm{fb}}(x) \leq J(x, \mathbf{y}^*, \mathbf{z}) + \delta, \qquad \mathbf{y}, \mathbf{z} \in \mathbf{H}.$$

A pair $(\mathbf{y}^*, \mathbf{z}^*)$ satisfying the above is said to be a δ-saddle point (for x).

Theorem 1.2. *Let Assumption 1.1 hold. Then $V^{\mathrm{fb}} = U^{\mathrm{fb}} = V = U = u$ on $\bar{G}$. Moreover, fix $x \in \bar{G}$ and a positive sequence $\delta_n \to 0$. Then, for each n, one can choose c^n and d^n in $(0, \infty)$, such that, with*

$$\mathbf{y}^n = (\bar{p}, c^n), \qquad \mathbf{z}^n = (-\bar{p}, d^n),$$

$(\mathbf{y}^n, \mathbf{z}^n)$ is a δ_n-saddle point. As a consequence, for each n, the corresponding process X^n is equal to the solution X of

$$dX_t = 2\bar{p}(X_t)dW_t, \qquad X_0 = x. \tag{1.12}$$

We end this section with a discussion on the possibility to extend our results to $\mathcal{C}^1$ solutions, based on the following.

Problem 1.1. *Let $u \in \mathcal{C}^1(\bar{G})$ be a viscosity solution to $\Delta_\infty = h$ on a bounded, smooth domain $G \subset \mathbb{R}^m$, where h is continuous, and $Du \neq 0$ in $\bar{G}$. Does there exists a sequence, u_n, of $\mathcal{C}^2$ functions converging to u in $W^{1,\infty}$, such that $\Delta_\infty u_n \to h$ uniformly in G?*

This is closely related to a problem posed in [8] (Section 8, problem (4)), and solved by Yu [13]: *Let $u : U \to \mathbb{R}$ be Lipschitz, let $h_1, h_2 : U \to \mathbb{R}$ be continuous on an open set $U \subset \mathbb{R}^m$, and assume that $\Delta_\infty u = h_1$ and $\Delta_\infty u = h_2$, both in the viscosity sense. Does it follow that $h_1 = h_2$?*

If Problem 1.1 is answered in the affirmative, it will be possible to extend Theorem 1.1 to $\mathcal{C}^1$ solutions. Indeed, given a $\mathcal{C}^1$ solution to (1.1) with non-vanishing gradient, one can define a sequence u_n of $\mathcal{C}^2$ approximations in $W^{1,\infty}$, which would then satisfy, classically, $-2\Delta_\infty = h_n$ in G and $u_n = g_n$ on ∂G, where $h_n \to h$ and $g_n \to g$ uniformly. Our results yield SDE's of the form $dX_t = 2\bar{p}_n(X_t)dW_t + 2r_n(X_t)dt$, where $\bar{p}_n = Du_n/\|Du_n\|$ and $r_n \cdot p_n = 0$ as limiting trajectories under a nearly optimal play for the game associated with (h_n, g_n). From this paper's technique, it is easy to deduce (using uniform estimates on the exit time such as (2.15)) that a near optimal play for the (h_n, g_n) game is also near optimal for the original (h, g) game. Since $\bar{p}_n \to \bar{p}$, and given $r \in R$ one can find r_n as above, converging to r, this would yield the SDE $dX_t = \bar{p}(X_t)dW_t + 2r(X_t)dt$ as a limiting trajectory for the (h, g) game, thus extending Theorem 1.1.

2. Proofs

The proofs of Theorem 1.1 and 1.2 appear in Subsections 2.1 and 2.2, respectively.

2.1. Near optimal play under the Elliott-Kalton SDG

In this subsection we prove Theorem 1.1 and so throughout this subsection the hypothesis of Theorem 1.1 will be in force and will not be stated explicitly in statements of results.

Recall that q is defined in (1.3). Most of the subsection will be concerned with proving the following special case of Theorem 1.1 in which $r = q$.

Theorem 2.1. *Theorem 1.1 is valid in the special case where $r = q$.*

The organization of this subsection is as follows. We begin by recalling the Bellman-Isaacs form of (1.1), which is given in (2.4). Proposition 2.1 analyzes near maximizing and minimizing variables in (2.4). Following the construction of a strategy-control pair (that is later slightly

modified, in the proof of Theorem 2.1), Proposition 2.2 proves its near optimality. Proposition 2.3 shows that, under this pair, the coefficients of the state process converge to those of (1.2). This result, along with Lemmas 2.2–2.4, is then used to prove weak convergence of the corresponding processes and exit times, and these results are combined to establish Theorem 2.1. The proofs of Propositions 2.1 and 2.3 appear near the end of the subsection. At the end of the subsection, we prove Theorem 1.1.

For $(a,c),(b,d) \in \mathcal{H}$, $p \in \mathbb{R}^m$ and $S \in \mathcal{S}(m)$ (the set of symmetric $m \times m$ matrices), let

$$\phi(a,b,c,d;p,S) = -\frac{1}{2}(a-b)'S(a-b) - (c+d)(a+b) \cdot p, \tag{2.1}$$

and denote

$$\Lambda^+(p,S) = \sup_{(b,d)\in\mathcal{H}} \inf_{(a,c)\in\mathcal{H}} \phi(a,b,c,d;p,S). \tag{2.2}$$

It has been shown in [3] (see Proposition 5.1 therein) that for every $p \in \mathbb{R}^m$, $p \neq 0$ and $S \in \mathcal{S}(m)$, one has

$$\Lambda^+(p,S) = \Lambda(p,S) := -2|p|^{-2}p'Sp. \tag{2.3}$$

Throughout, we denote

$$p(x) = Du(x), \quad \bar{p}(x) = \frac{p(x)}{|p(x)|}, \quad S(x) = D^2u(x),$$

$$q(x) = \frac{1}{|p(x)|^2}(D^2u(x)\,Du(x) - \Delta_\infty u(x)\,Du(x))$$

and

$$\psi(x,y,z) = -h(x) + \phi(a,b,c,d;p(x),S(x)), \qquad y=(a,c),\ z=(b,d).$$

Since u satisfies (1.1) in the classical sense, and since $\Lambda^+ = \Lambda$, we have

$$\sup_{z\in\mathcal{H}}\inf_{y\in\mathcal{H}} \psi(x,y,z) = 0, \qquad x \in \bar{G}. \tag{2.4}$$

Identity (2.4) will be the basis for the construction of a δ-optimal play for the lower game. To present the construction we first need the following result. Its proof appears at the end of the subsection.

Proposition 2.1. *For every $\delta \in (0,\infty)$ there exist $d^\delta \in (0,\infty)$ and $a^\delta : \bar{G} \to \mathcal{S}^{m-1}$ such that the following holds.*

  i. *For $x \in \bar{G}$, let $z^\delta(x) \equiv (b^\delta(x), d^\delta(x)) = (-\bar{p}(x), d^\delta)$. Then*

$$\inf_{y\in\mathcal{H}} \psi(x,y,z^\delta(x)) = \min_{y\in\mathcal{S}^{m-1}\times\{0\}} \psi(x,y,z^\delta(x)) \in [-\delta, 0]. \tag{2.5}$$

  Moreover, $d^\delta \to \infty$ as $\delta \to 0$.

ii. With $y^\delta(x) = (a^\delta(x), 0)$,

$$\psi(x, y^\delta(x), z^\delta(x)) \in [-\delta, \delta], \quad x \in \bar{G}. \tag{2.6}$$

Moreover, a^δ is Lipschitz in x for every δ. Finally,

$$a^\delta \to \bar{p}, \quad uniformly, \ as \ \delta \to 0, \tag{2.7}$$

and

$$d^\delta(a^\delta - \bar{p}) \to 2q, \quad uniformly, \ as \ \delta \to 0. \tag{2.8}$$

To define $\beta^{x,\delta} \equiv \beta^\delta$ (the dependence on the initial condition is suppressed in some instances), let $Y = (Y^0, K) \in M$, with $Y^0 = (A, C)$, be given, and consider the equation

$$dX = (A - b^\delta(X))dW + (C + d^\delta(X))(A + b^\delta(X))ds, \ X_0 = x \tag{2.9}$$

where $b^\delta(x) = -\bar{p}(x)$, and $d^\delta(x) = d^\delta$. By the Lipschitz property of b^δ, this equation has a unique solution. This defines a process $Z^\delta = (b^\delta(X), d^\delta(X))$, hence a mapping, $Y \mapsto (Z^\delta, d^\delta) \in M$, which is easily seen to be a strategy. This strategy will be denoted by β^δ.

Next, consider the equation

$$dX = P^\delta(X)dW + Q^\delta(X)ds, \ X_0 = x \tag{2.10}$$

where

$$P^\delta(x) = a^\delta(x) - b^\delta(x) = a^\delta(x) + \bar{p}(x),$$

$$Q^\delta(x) = (c^\delta(x) + d^\delta(x))(a^\delta(x) + b^\delta(x)) = d^\delta(a^\delta(x) - \bar{p}(x)), \ c^\delta = 0.$$

Since the coefficients P^δ, Q^δ are Lipschitz, there is a unique solution to (2.10). Define $\bar{Y}^{\delta,x} = \bar{Y}^\delta = (a^\delta(X), c^\delta(X))$. Clearly $(\bar{Y}^\delta, 1) \in M$ and $\beta^\delta(\bar{Y}^\delta, 1) = ((b^\delta(X), d^\delta), d^\delta)$.

Towards arguing that the strategy-control pair constructed above forms a nearly optimal play, we shall use the following

Lemma 2.1. *For every $x \in \bar{G}$, $Y, Z \in M$, one has*

$$u(x) = \boldsymbol{E}\Big[u(X_{t \wedge \tau}) + \int_0^{t \wedge \tau} (\psi(X_s, Y_s, Z_s) + h(X_s))ds\Big], \qquad t \geq 0, \tag{2.11}$$

and, if $\boldsymbol{E}[\tau] < \infty$, one has

$$J(x, Y, Z) = V(x) - \boldsymbol{E}\Big[\int_0^\tau \psi(X_s, Y_s, Z_s)ds\Big], \tag{2.12}$$

where $X = X(x, Y, Z)$ and $\tau = \tau_G(X)$.

Proof. The two identities are immediate consequences of Ito's formula applied to the smooth function u, the boundary condition $u = g$ on ∂G, and the equality $u = V$. $\qquad\square$

In what follows, let $c_0 < \infty$ be a constant such that

$$|h(x)| + |g(y)| + |u(x)| + |Du(x)| + \text{Lip}(\bar{p}) \le c_0, \qquad x \in \bar{G}, \ y \in \partial G. \tag{2.13}$$

Denote $\underline{h} = \inf_{x \in G} |h(x)|$.

Proposition 2.2. *Fix $x \in \bar{G}$. There exist $\eta, c \in (0, \infty)$ such that for every $\delta \in (0, \eta)$, $(\bar{Y}^{x,\delta}, \beta^{x,\delta})$ forms a $c\delta$-optimal play for the initial condition x.*

Proof. Fix $Y = (Y^0, K) \in M$ with $Y^0 = (A, C)$. Let X denote the unique solution of (2.9) with this choice of (A, C) and let Z^δ, d^δ be as introduced above (2.10). Then $\beta^\delta(Y) = (Z^\delta, d^\delta)$. By (2.5), for every s,

$$\psi(X_s, Y_s^0, Z_s^\delta) = \psi(X_s, Y_s^0, z^\delta(X_s)) \ge -\delta. \tag{2.14}$$

Let $\eta = \underline{h}/2$. Consider first the case $h > 0$. For $\delta < \eta$, we have by (2.11)

$$\boldsymbol{E}[t \wedge \tau] \le c_1 := 4\underline{h}^{-1}c_0, \tag{2.15}$$

and consequently $\boldsymbol{E}[\tau] \le c_1$, where $\tau = \tau^x[Y, \beta^\delta]$. Hence using (2.14) in (2.12),

$$J^x(Y, \beta^\delta) \le V(x) + \delta \boldsymbol{E}[\tau] \le V(x) + c_1\delta. \tag{2.16}$$

Since $Y \in M$ is arbitrary, this shows that β^δ is $c_1\delta$-optimal.

Consider now the case $h < 0$. Fix $e \in \mathcal{S}^{m-1}$ and let $(e, 1) = \widetilde{Y} \in M$. It is easily checked (see Lemma 3.1 of [3]) that $\inf_{\beta \in \Gamma} J^x(\widetilde{Y}, \beta) > -\infty$. Thus $\inf_\delta C(\beta^\delta) := \underline{c} > -\infty$, where for $\beta \in \Gamma$, $C(\beta) = \sup_{Y \in M} J^x(Y, \beta)$. Let $M_\delta = \{Y \in M : J^x(Y, \beta^\delta) > \underline{c} - 1\}$. Then $C(\beta) = \sup_{Y \in M_\delta} J^x(Y, \beta)$. Note that for $Y \in M_\delta$, $\tau = \tau^x(Y, \beta^\delta) < \infty$ a.s. and

$$\underline{c} - 1 < J^x(Y, \beta^\delta) \le -\underline{h}\boldsymbol{E}[\tau] + c_0.$$

Thus for the case $h < 0$ as well, β^δ is $c_2\delta$-optimal, for some $c_2 \in (0, \infty)$.

Recall that $\bar{Y}_s^\delta = y^\delta(X_s)$, where X is the unique solution of (2.10) and note that

$$\beta^\delta((\bar{Y}^\delta, 1)) = (Z^\delta(X), d^\delta).$$

By (2.6),

$$\psi(X_s, y^\delta(X_s), z^\delta(X_s)) \le \delta. \tag{2.17}$$

Observing that $\boldsymbol{E}[\tau] \le c_1$, where $\tau = \tau^x(\bar{Y}^\delta, \beta^\delta)$, we have using (2.17) in (2.12),

$$J^x(\bar{Y}^\delta, \beta^\delta) \ge V(x) - c_1\delta \ge \sup_{Y \in M} J^x(Y, \beta^\delta) - 2(c_1 \vee c_2)\delta, \tag{2.18}$$

where the last inequality follows from the $(c_1 \vee c_2)\delta$-optimality of β^δ. The result follows. $\qquad \square$

The proof of the following proposition is given towards the end of the subsection. Denote by p^* and q^* the continuous, bounded extensions of $\bar{p}$ and q to $\mathbb{R}^m$, satisfying the hypotheses of Theorem 2.1.

Proposition 2.3. *Let $\{\delta_n\}_{n\geq 1}$ be a sequence in $\mathbb{R}_+$ such that $\delta_n \to 0$ as $n \to \infty$. Then there exists a sequence of (open) domains $G_{n-1} \subset\subset G_n \subset G$, $G_n \uparrow G$ as $n \to \infty$ and continuous, uniformly bounded maps p_n^*, q_n^* from $\mathbb{R}^m$ to itself, $p_n^* \to p^*$, $q_n^* \to q^*$, uniformly on $\mathbb{R}^m$, such that $p_n^* = \bar{p}_n$ and $q_n^* = \bar{q}_n$ on G_n, where $\bar{p}_n = \frac{1}{2}(a^{\delta_n} + \bar{p})$ and $\bar{q}_n = \frac{1}{2}d^{\delta_n}(a^{\delta_n} - \bar{p})$.*

Lemma 2.2. *With notation as in Proposition 2.3, let $\bar{X}_n, \bar{X}$ be solutions of*

$$d\bar{X}_n = 2p_n^*(\bar{X}_n)dW + 2q_n^*(\bar{X}_n)dt, \qquad d\bar{X} = 2p^*(\bar{X})dW + 2q^*(\bar{X})dt,$$

respectively, starting from x, and given on suitable filtered probability spaces. Denote

$$\bar{\tau}(n,k) = \tau_{G_k}(\bar{X}^n), \quad \bar{\tau}(k) = \tau_{G_k}(\bar{X}), \quad \bar{\tau} = \tau_G(\bar{X}).$$

Then there exists a sequence $\{\ell_n\}_{n\geq 1}$, $\ell_n \uparrow \infty$ as $n \to \infty$, such that

$$(\bar{X}_n(\cdot \wedge \bar{\sigma}_n), \bar{\sigma}_n) \Rightarrow (\bar{X}(\cdot \wedge \bar{\tau}), \bar{\tau})$$

as a sequence of $C([0,\infty) : \bar{G}) \times [0,\infty]$ -valued random variables, where $\bar{\sigma}_n = \bar{\tau}(n,\ell_n)$.

Proof. The coefficients p_n^* and q_n^* converge uniformly on $\mathbb{R}^m$ to p^* and q^*, respectively, by Proposition 2.3. Moreover, by assumption, weak uniqueness holds for solutions to the SDE associated with (p^*, q^*), starting from x, for any $x \in \mathbb{R}^m$. Thus Theorem 11.1.4 of [11] is in force, and we can deduce that $\bar{X}^n$ converges to $\bar{X}$ in distribution, as $n \to \infty$.

We now assume without loss of generality that $\bar{X}_n, \bar{X}$ are given on a common probability space and $\bar{X}_n \to \bar{X}$, a.s., in $C([0,\infty))$. For $t > 0$ let $E_t = \{\omega : \bar{\tau}(\omega) \leq t\}$. Fix $\omega \in E_t$. Given $k \in \mathbb{N}$, choose $\delta > 0$ such that $|y_1 - y_2| > \delta$ for all $y_1 \in G_k$, $y_2 \in \partial G$. Let $n_0 = n_0(\delta, t, \omega)$ be such that, for all $n \geq n_0$, $|\bar{X}_n - \bar{X}|_t^* < \delta$. Note that $\bar{X}(\bar{\tau}(\omega)) \in \partial G$ and so $\bar{X}_n(\bar{\tau}(\omega)) \notin G_k$. In particular, $\bar{\tau}(n,k)(\omega) \leq \bar{\tau}(\omega)$. Letting $n \to \infty$, we get $\limsup_{n\to\infty} \bar{\tau}(n,k) \leq \bar{\tau}$, for all $\omega \in E_t$. Since $t > 0$ is arbitrary, we have that for every $k \in \mathbb{N}$, $\limsup_{n\to\infty} \bar{\tau}(n,k) \leq \bar{\tau}$ a.s. Using lower semi-continuity property of exit times we then have a.s.,

$$\bar{\tau}(k) \leq \liminf_{n\to\infty} \bar{\tau}(n,k) \leq \limsup_{n\to\infty} \bar{\tau}(n,k) \leq \bar{\tau}.$$

Also note that $\bar{\tau}(k) \to \bar{\tau}$ a.s., as $k \to \infty$.

Let $F = \{\bar{\tau} < \infty\}$. In what follows, for an event E, we will write $\boldsymbol{P}(EF)$ as $\boldsymbol{P}_F(E)$. $\boldsymbol{P}_{F^c}$ is defined similarly. From the above display we have that for every $\varepsilon > 0$

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \boldsymbol{P}_F(|\bar{\tau}(n,k) - \bar{\tau}| > \varepsilon) = 0.$$

We can then find a sequence $\{\varepsilon(k)\}_{k\geq 1}$, $\varepsilon(k) \in (0,\infty)$ such that $\varepsilon(k) \downarrow 0$ as $k \to \infty$ and

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \boldsymbol{P}_F(|\bar{\tau}(n,k) - \bar{\tau}| > \varepsilon(k)) = 0.$$

Finally, choose a sequence $\{\ell_n\}_{n\geq 1}$ such that $\ell_n \uparrow \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \boldsymbol{P}_F(|\bar{\tau}(n,\ell_n) - \bar{\tau}| > \varepsilon(\ell_n)) = 0.$$

In a similar fashion, by choosing a further subsequence if needed, we have that for every $r > 0$

$$\lim_{n \to \infty} \boldsymbol{P}_{F^c}(\bar{\tau}(n, \ell_n) \leq r) = 0.$$

Combining the above displays we have $\bar{\sigma}_n = \bar{\tau}(n, \ell_n) \to \bar{\tau}$ in probability as $n \to \infty$. The result follows. $\qquad \square$

Lemma 2.3. *Let X^n be the (pathwise) unique solution of* (2.10) *with $\delta = \delta_n$ (stopped when the boundary is reached). Let $(G_n, p_n^*, q_n^*)_{n \geq 1}$, p^*, q^* be as in Proposition 2.3 and $\{\ell_n\}_{n \geq 1}$ be as in Lemma 2.2. Let X solve* (1.2) *with initial condition x. Then*

$$(X^n(\cdot \wedge \eta_n), \eta_n) \Rightarrow (X(\cdot \wedge \tau), \tau),$$

where $\eta^n = \tau_{G^n \wedge \ell_n}(X^n)$ and $\tau = \tau_G(X)$.

Proof. Let $\bar{X}^n, \bar{X}$ be as in Lemma 2.2. Then from Proposition 2.3 $(\bar{X}(\cdot \wedge \bar{\tau}), \bar{\tau})$ has the same law as $(X(\cdot \wedge \tau), \tau)$ and $(\bar{X}^n(\cdot \wedge \bar{\eta}_n), \bar{\eta}_n)$ has the same law as $(X^n(\cdot \wedge \eta_n), \eta_n)$, where $\bar{\eta}^n$ is defined similarly to η^n by replacing X^n with $\bar{X}^n$. By lower semi-continuity of exit times, $\liminf_n \bar{\tau}(n, n) \geq \bar{\tau}$ a.s. The result now follows from Lemma 2.2 on noting that $\bar{\eta}^n = \bar{\tau}(n, n) \wedge \bar{\sigma}^n$.

Lemma 2.4. *Let X be a solution of* (1.2) *given on some filtered probability space, with $X_0 = x \in \bar{G}$. Let $\tau = \tau_G(X)$. Then $\boldsymbol{E}[\tau] < \infty$ and*

$$u(x) = \boldsymbol{E}\left[g(X_\tau) + \int_0^\tau h(X_s)ds\right].$$

Proof. Applying Itô's formula to $u(X)$ and recalling that u is a classical solution of (1.1), we obtain

$$u(x) = \boldsymbol{E}\left[u(X_{\tau \wedge t}) + \int_0^{\tau \wedge t} h(X_s)ds\right],$$

for every $t > 0$. The property $\boldsymbol{E}[\tau] < \infty$ is now immediate on recalling that h is either positive or negative, and bounded away from zero. The result follows on sending $t \to \infty$. $\qquad \square$

Proof of Theorem 2.1. We will only prove the statement concerning the lower game. The proof for the upper game is analogous. Fix $x \in \bar{G}$. Let $\{X^n, G_n, p_n^*, q_n^*, \ell_n, \eta_n\}$ be as in Lemma 2.3. Let $(\bar{Y}^n, \beta^n) = (\bar{Y}^{x, \delta_n}, \beta^{x, \delta_n})$ where, for $\delta > 0$, $(\bar{Y}^{x, \delta}, \beta^{x, \delta})$ is as in Proposition 2.2. Note that $X^n = X^x(\bar{Y}^n, \beta^n)$. We assume, without loss of generality, that $\delta_n < \underline{h}/2$ for $n \geq 1$. Then, as in the proof of Proposition 2.2, we deduce that

$$\boldsymbol{E}(\eta_n) \leq \boldsymbol{E}(\tau_G(X^n)) \leq 4\underline{h}^{-1}c_0, \qquad (2.19)$$

where c_0 was introduced in (2.13). From Lemma 3.2 of [3], there exist $\widetilde{Y}^n \in M$ and $\{\delta_n^1\}_{n \geq 1}$, $\delta_n^1 \downarrow 0$, such that $\widetilde{Y}_{t \wedge \eta_n}^n = \bar{Y}_{t \wedge \eta_n}^n$ and

$$\boldsymbol{E}\{\widetilde{\tau}_n - \eta_n \,|\, \mathcal{F}_{\eta_n}\} \leq \delta_n^1, \ \ \boldsymbol{E}\{|\widetilde{X}^n - X^n(\eta_n)|_*^2 | \mathcal{F}_{\eta_n}\} \leq \delta_n^1, \qquad (2.20)$$

where $\widetilde{X}^n = X^x(\widetilde{Y}^n, \beta^n)$, $\widetilde{\tau}^n = \tau^x(\widetilde{Y}^n, \beta^n)$ and $|\widetilde{X}^n - X^n(\eta_n)|_* = \sup_{t \in [\eta_n, \widetilde{\tau}_n]} |\widetilde{X}^n(t) - X^n(\eta_n)|$.

Recall that β^n is $c\delta_n$-optimal. We now show that $\widetilde{Y}^n$ is δ_n^*-optimal for play against β^n, for some sequence $\delta_n^* \to 0$. From Lemma 2.3 and (2.19) we have that

$$\left| \boldsymbol{E}\Big[\int_0^{\eta_n} h(X^n(s))ds + V(X_{\eta_n}^n) \Big] - \boldsymbol{E}\Big[\int_0^{\tau} h(X(s))ds + V(X_\tau) \Big] \right| = \delta_n^2 \to 0, \text{ as } n \to \infty.$$

From (2.20)

$$\left| \boldsymbol{E}\Big[\int_0^{\widetilde{\tau}_n} h(\widetilde{X}^n(s))ds + V(\widetilde{X}_{\widetilde{\tau}_n}^n) \Big] - \boldsymbol{E}\Big[\int_0^{\eta_n} h(X^n(s))ds + V(X_{\eta_n}^n) \Big] \right| = \delta_n^3 \to 0, \text{ as } n \to \infty.$$

Setting $\delta_n^* = \delta_n^2 + \delta_n^3 + c\delta_n$, we have on combining the above two displays

$$
\begin{aligned}
J^x(\widetilde{Y}^n, \beta^n) &= \boldsymbol{E}\Big[\int_0^{\widetilde{\tau}_n} h(\widetilde{X}_s^n)ds + g(\widetilde{X}^n(\widetilde{\tau}_n)) \Big] \\
&\geq \boldsymbol{E}\Big[\int_0^{\tau} h(X_s)ds + V(X(\tau)) \Big] - (\delta_n^2 + \delta_n^3) \\
&= V(x) - (\delta_n^2 + \delta_n^3) \\
&\geq \sup_{Y \in M} J^x(Y, \beta^n) - \delta_n^*,
\end{aligned}
$$

where the equality in the third line above follows from Lemma 2.4 and the last inequality is a consequence of $c\delta_n$-optimality of β^n. Finally, from (2.20) $\sup_{0 \leq t < \infty} |X^n(t \wedge \eta_n) - \widetilde{X}^n(t \wedge \widetilde{\tau}_n)| \to 0$ and $|\eta_n - \widetilde{\tau}_n| \to 0$ in probability as $n \to \infty$. Thus, from Lemma 2.3 $(\widetilde{X}^n(\cdot \wedge \widetilde{\tau}_n), \widetilde{\tau}_n) \Rightarrow (X(\cdot \wedge \tau), \tau)$ and the result follows. $\qquad\square$

Proof of Proposition 2.3. For $d \in \mathbb{N}$, let $f_n, f : \bar{G} \to \mathbb{R}^d$ be uniformly bounded continuous maps such that $f_n \to f$ uniformly on $\bar{G}$. Let $F : \mathbb{R}^m \to \mathbb{R}^d$ be a uniformly continuous bounded extension of f. Consider a sequence $\{E_n\}$ of (open) domains with $E_{n-1} \subset\subset E_n \subset G$, $E_n \uparrow G$ as $n \to \infty$. We will show that there is a collection of uniformly bounded, continuous maps $\{F_n^k : n \geq 1, k \geq 1\}$ such that F_n^k agrees with f_n on E_k and along some subsequence $\{k_n\}_{n \geq 1}$, $F_n^{k_n} \to F$, uniformly on $\mathbb{R}^m$. The result will then follow, on setting $F = (p^*, q^*)$, $f = (\bar{p}, q)$, $f_n = (\bar{p}_n, \bar{q}_n)'$, with $G_n = E_{k_n}$.

Define
$$\widetilde{F}_n(x) = f_n(x)1_{x \in \bar{G}} + F(x)1_{x \in \mathbb{R}^m \setminus \bar{G}}.$$

Let ψ be a C^∞ function on $\mathbb{R}^d$ such that $0 \leq \psi(x) \leq 1$, $\text{supp}(\psi) \subset B_1(0)$ and $\int_{\mathbb{R}^d} \psi(x)dx = 1$, where $B_r(0)$ is the ball of radius r in $\mathbb{R}^m$, centered at 0. Let $\psi_k(x) = k^m \psi(kx)$. Define

$$\bar{F}_n^k(x) = \int_{\mathbb{R}^m} \widetilde{F}_n(x - y)\psi_k(y)dy, \quad \bar{F}^k(x) = \int_{\mathbb{R}^m} F(x - y)\psi_k(y)dy, \quad x \in \mathbb{R}^m.$$

Let $\rho^k \in C^\infty(\mathbb{R}^m)$ be such that $0 \leq \rho^k(x) \leq 1$ and

$$\rho^k(x) = \begin{cases} 1 & \text{if } x \in E_k \\ 0 & \text{if } x \in G^c. \end{cases}$$

Define

$$F_n^k(x) = \rho^k(x)f_n(x) + (1 - \rho^k(x))\bar{F}_n^k(x), \quad x \in \mathbb{R}^m$$

and

$$F^k(x) = \rho^k(x)F(x) + (1 - \rho^k(x))\bar{F}^k(x), \quad x \in \mathbb{R}^m.$$

Note that $F_n^k(x) = f_n(x)$ and $F^k(x) = f(x)$ for $x \in E_k$ and

$$F_n^k(x) - F^k(x) = \rho^k(x)(f_n(x) - f(x)) + (1 - \rho^k(x))(\bar{F}_n^k(x) - \bar{F}^k(x)), \quad x \in \mathbb{R}^m.$$

Also

$$\sup_{x \in \mathbb{R}^m} \sup_{k \geq 1} |\bar{F}_n^k(x) - \bar{F}^k(x)| \leq \sup_{x \in \bar{G}} |f_n(x) - f(x)| \to 0, \quad \text{as } n \to \infty.$$

Combining the above two displays

$$\sup_{x \in \mathbb{R}^m} \sup_{k \geq 1} |F_n^k(x) - F^k(x)| \to 0, \quad \text{as } n \to \infty.$$

Next note that $\sup_{x \in \mathbb{R}^m} |\bar{F}^k(x) - F(x)| \to 0$, as $k \to \infty$ and therefore

$$\sup_{x \in \mathbb{R}^m} |F^k(x) - F(x)| \to 0, \quad \text{as } k \to \infty.$$

Using the above two displays, we can find a sequence $\{k_n\}$ such that $F_n^{k_n} \to F$ uniformly on $\mathbb{R}^m$. By construction F_n^k agrees with f_n on E_k. The result follows. $\square$

Proof of Proposition 2.1. We begin by constructing functions a^δ for which all conclusions of the proposition hold, save the Lipschitz property. We will then argue that one can find a Lipschitz regularization of each a^δ, for which all conclusions are still valid.

With $b = b(x) = -\bar{p}(x)$, the second term of (2.1) takes the form $(c+d)(1 - a \cdot \bar{p}(x))|p(x)| \geq 0$, and therefore the infimum of $\psi(x, (a, c), (b(x), d))$, over c, is attained at $c = 0$. The function $a \mapsto \psi(x, (a, 0), (-\bar{p}(x), d))$ is continuous, and thus the minimum over $\mathcal{S}^{m-1}$ is attained. For an arbitrary choice of d^δ, we have by (2.4),

$$\gamma^\delta(x) := \min_{y \in \mathcal{S}^{m-1} \times \{0\}} \psi(x, y, z^\delta(x)) \leq 0. \tag{2.21}$$

Later in the proof it is shown that for a suitable choice of d^δ, $\gamma^\delta(x) \geq -\delta$ for all $x \in \bar{G}$.

For each δ and x let

$$a^\delta(x) \text{ be a minimizer of } a \mapsto \psi(x, (a, 0), (-\bar{p}(x), d^\delta)) \text{ over } \mathcal{S}^{m-1}. \tag{2.22}$$

Write $y^\delta(x) = (a^\delta(x), 0)$. From (2.21) $\gamma^\delta(x) = \psi(x, y^\delta(x), z^\delta(x)) \leq 0$.

We show now that for any choice of d^δ such that $d^\delta \to \infty$ as $\delta \to 0$,

$$a^\delta(x) \to \bar{p}(x) \text{ as } \delta \to 0, \text{ uniformly in } x. \tag{2.23}$$

Assuming the contrary, there exists $\varepsilon > 0$ and, for every $\delta > 0$, $x_\delta \in \bar{G}$, such that

$$|a^\delta(x_\delta) - \bar{p}(x_\delta)| > \varepsilon. \tag{2.24}$$

However, because of the upper bound on γ^δ, it follows that

$$d^\delta(1 - a^\delta(x) \cdot \bar{p}(x))|p(x)| \le c_1,$$

for some constant c_1 not depending on x and δ. This contradicts (2.24) and thus (2.23) follows. Henceforth we will assume that $d^\delta \to \infty$ as $\delta \to 0$.

Since y^δ is a minimizer, we have that $\psi(x, y^\delta(x), z^\delta(x)) \le \psi(x, (\bar{p}(x), 0), z^\delta(x))$. Along with the uniform convergence in (2.23), this implies

$$\limsup_{\delta \to 0} \sup_x d^\delta(1 - a^\delta(x) \cdot \bar{p}(x))|p(x)| \le 0.$$

Consequently,

$$d^\delta(1 - a^\delta(x) \cdot \bar{p}(x)) \to 0 \text{ as } \delta \to 0, \text{ uniformly in } x. \tag{2.25}$$

We next show that

$$Q^\delta(x) := d^\delta(a^\delta(x) - \bar{p}(x)) \to 2q(x) \text{ as } \delta \to 0, \text{ uniformly in } x. \tag{2.26}$$

Denote by $\widetilde{\phi}$ the map $a \mapsto \phi(a, b(x), 0, d^\delta; p(x), S(x))$. By the Lagrange multipliers theorem, every $a \in \mathcal{S}^{m-1}$, which minimizes $\widetilde{\phi}(a)$ satisfies $D\widetilde{\phi}(a) + \lambda a = 0$ for some $\lambda \in \mathbb{R}$. Thus by definition of $a^\delta(x)$, suppressing the dependence on δ and x,

$$\lambda a = S(a + \bar{p}) + dp, \tag{2.27}$$

$$\lambda = a'S(a + \bar{p}) + da \cdot p. \tag{2.28}$$

Hence

$$Q = d(a - \bar{p}) = da - \frac{dp}{|p|}$$

$$= da - \frac{a}{|p|}\lambda + \frac{S(a + \bar{p})}{|p|}$$

$$= d(1 - a \cdot \bar{p})a - \frac{a}{|p|}a'S(a + \bar{p}) + \frac{S(a + \bar{p})}{|p|} \to -\frac{2\bar{p}}{|p|}\bar{p}'S\bar{p} + \frac{2}{|p|}S\bar{p} = 2q,$$

where the convergence is uniform, and we have used (2.23) and (2.25) on the last line. This shows (2.26).

We now estimate γ^δ. Suppressing x and δ,

$$\gamma = \psi(x, y, z) = -h - \frac{1}{2}(a + \bar{p})'S(a + \bar{p}) - d(a - \bar{p}) \cdot p.$$

where the 'o' notation corresponds to uniform convergence as $\delta \to 0$. Let us show that the conclusions of Proposition 2.1, analogous to (2.6), (2.7) and (2.8), continue to hold with a^δ replaced by $\bar{a}^\delta$ and q by r. That is, keeping the notation of that proposition, writing $\bar{y}^\delta(x) = (\bar{a}^\delta(x), 0)$, we claim that

$$\psi(x, \bar{y}^\delta(x), z^\delta(x)) \in [-\delta, \delta], \quad x \in \bar{G}. \tag{2.31}$$

$$\bar{a}^\delta \to \bar{p}, \quad \text{uniformly, as } \delta \to 0, \tag{2.32}$$

and

$$d^\delta(\bar{a}^\delta - \bar{p}) \to 2r, \quad \text{uniformly, as } \delta \to 0. \tag{2.33}$$

Indeed, using (2.30) and the fact $r \cdot p = 0$ in the definition of ψ, we have $\psi(x, \bar{y}^\delta(x), z^\delta(x)) - \psi(x, y^\delta(x), z^\delta(x)) \to 0$ uniformly. Thus by (2.6), $\psi(x, \bar{y}^\delta(x), z^\delta(x)) \to 0$ uniformly, and (2.31) holds up to re-parameterization. Next, (2.32) and (2.33) are immediate from (2.30).

Modifying definition of $\bar{Y}^\delta$ (given below (2.10)) by replacing a^δ by $\bar{a}^\delta$, it is easily checked that Proposition 2.2 holds. Proposition 2.3 and Lemmas 2.2 and 2.3 continue to hold when definitions of $q^*, \bar{p}_n, \bar{q}_n, p_n^*, q_n^*$ are modified by replacing a^δ with $\bar{a}^\delta$ and q by r. Also note that since $r \cdot p = 0$, Lemma 2.4 holds when X is given as a solution of (1.4) rather than (1.2). With these observations, the proof of Theorem 1.1 is completed exactly as that of Theorem 2.1. $\square$

2.2. Saddle point under the feedback control formulation

In this subsection we prove Theorem 1.2. Fix $\delta > 0$ and $x \in \bar{G}$. Let $\mathbf{z}^\delta \in \mathbf{H}$ be as in Proposition 2.1(i). Let $\mathbf{y} = (a, c) \in \mathbf{H}$ and let $\bar{X}$ be the unique solution of

$$d\bar{X} = \bar{P}(\bar{X})dW + \bar{Q}(\bar{X})ds, \ \bar{X}_0 = x,$$

where $\bar{P} = a - b^\delta = a + \bar{p}$ and $\bar{Q} = (c + d^\delta)(a + b^\delta) = (c + d^\delta)(a - \bar{p})$. Let $\bar{Y} = (a(\bar{X}), c(\bar{X}))$ and $S = \sup_{x \in \bar{G}} c(x)$. Then $Y = (\bar{Y}, S) \in M$ and from (2.16)

$$J(x, \mathbf{y}, \mathbf{z}^\delta) \le V(x) + c_1\delta.$$

Thus

$$U^{\text{fb}}(x) \le \sup_{\mathbf{y} \in \mathbf{H}} J(x, \mathbf{y}, \mathbf{z}^\delta) \le V(x) + c_1\delta \tag{2.34}$$

and consequently $U^{\text{fb}} \le V$.

In a manner analogous to the proof of Proposition 2.1, starting with $\tilde{a}^\delta = \bar{p}$, one can find $\tilde{c}^\delta \in (0, \infty)$ such that for some $c_2 \in (0, \infty)$ and all $\mathbf{z} = (b, d) \in \mathbf{H}$, $J(x, \mathbf{y}^\delta, \mathbf{z}) \ge U(x) - c_2\delta$, where $\mathbf{y}^\delta = (\tilde{a}^\delta, \tilde{c}^\delta)$. Consequently

$$V^{\text{fb}}(x) \ge \inf_{\mathbf{z} \in \mathbf{H}} J(x, \mathbf{y}^\delta, \mathbf{z}) \ge U(x) - c_2\delta. \tag{2.35}$$

This shows $V^{\text{fb}} \ge U$. Since $V^{\text{fb}} \le U^{\text{fb}}$ and $U = V = u$, we have the first assertion in the Theorem. Finally combining (2.34) and (2.35), we have for all δ sufficiently small and all $(\mathbf{y}, \mathbf{z}) \in \mathbf{H}^2$

$$J(x, \mathbf{y}, \mathbf{z}^\delta) - c_1\delta \le V(x) = U(x) \le J(x, \mathbf{y}^\delta, \mathbf{z}) + c_2\delta.$$

The result follows. □

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