Nuclear Engineering and Technology 51 (2019) 1616–1625

Contents lists available at ScienceDirect

Nuclear Engineering and Technology

journal homepage: www.elsevier.com/locate/net

Original Article

High rate diffusion-scale approximation for counters with extendable dead time

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ARTICLE INFO

Article history:
Received 12 December 2018
Received in revised form
3 April 2019
Accepted 16 April 2019
Available online 3 May 2019

Keywords:
Dead time
Renewal process
Type II counters
Pair coefficient function

ABSTRACT

Measuring occurrence times of random events, aimed to determine the statistical properties of the governing stochastic process, is a basic topic in science and engineering, and has been the subject of numerous mathematical modeling approaches. Often, true statistical properties deviate from measured properties due to the so called *dead time* phenomenon, where for a certain time period following detection, the detection system is not operational. Understanding the dead time effect is especially important in radiation measurements, often characterized by high count rates and a non-reducible detector dead time (originating in the physics of particle detection). The effect of dead time can be interpreted as a suitable rarefied sequence of the original time sequence.

This paper provides a limit theorem for a high rate (diffusion-scale) counter with extendable (Type II) dead time, where the underlying counting process is a renewal process with finite second moment for the inter-event distribution. The results are very general, in the sense that they refer to a general inter arrival time and a random dead time with general distribution.

Following the theoretical results, we will demonstrate the applicability of the results in three applications: serially connected components, multiplicity counting and measurements of aerosol spatial distribution.

1. Introduction

1.1. General introduction

Counting and measuring occurrence times of random events in order to estimate statistical properties of underlying stochastic processes is a basic topic in science and engineering, and has been the subject of numerous mathematical modeling approaches. One of the most prominent problems in high rate measurement is the *dead time* phenomenon, where the sensing device takes a certain time to recover after each detection, during which the counter is not functional. The effect of dead time on the detection signal can be interpreted as the process of rarefying the sequence of occurrence times, by removing events within the dead time period following a previous event. One distinguishes two types of dead time models [1], In a type I counter, also referred to as non-extendable dead time, only counts that are within a dead time period following an actual detection are lost. Thus, for an event to inflict a dead time, it must appear in the rarefied sequence. In a type II counter, also called an extendable dead time, counts within a dead time period following an original event (detected or not) are lost. Thus, all events in the original sequence inflict a dead time. In nuclear engineering, type II counters are sometimes referred to as a paralyzing dead time, due to the fact that for sufficiently high count rates, the counter will be totally saturated, and the count rate will drop to 0. The effect of a type II counter, that is the topic of this work, may be described by considering a sequence $\mathscr{L}$ of random variables $0 \leq t_1 < t_2 < \cdots$, where t_n gives the occurrence time of the nth event after time 0. Let $\tau_{n+1} \geq 0$ denote the dead time following the nth detection (the duration of the dead time might also be random). The rarefied sequence is obtained from $\mathscr{L}$ by

$$\mathscr{L}' = \{t_1\} \cap \{t_i \in \mathscr{L} | t_i - t_{i-1} \geq \tau_i, i \geq 2\}. \quad (1)$$

Equation (1) is simply a formal way of removing all detections

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https://doi.org/10.1016/j.net.2019.04.015
1738-5733/

for which the waiting time from the previous detection is less than the dead time. We associate a counting process with each of the sequences $\mathscr{L}$ and $\mathscr{L}'$. Namely,[1] $R(t) = \#\{i : t_i \in \mathscr{L}, t_i \le t\}$ and $R'(t) = \#\{i : t_i \in \mathscr{L}', t_i \le t\}$. The statistical properties of $R'(t)$ are of interest. It is clear that they depend not only on the count rate, but on the statistical properties of $R(t)$; this is true even for the average count, and certainly for the higher moments. If the waiting times between consecutive events (or *inter-arrival time*) in $\mathscr{L}$ form a sequence of Independent and Identically Distributed (IID) random variables, then the counting process $R(t)$ is referred to as a **renewal process** [2]. The theory of renewal processes is a well developed theory with numerous application, and is a basic concept in the present study.

The practical significance of the dead time phenomenon has been widely recognized. It is observed in all types of measurement systems provided that the count rate is sufficiently high. Since high detection rates are very often in radiation measurements, there is a wide treatment of the subject in radiation literature [3–5] (to state a few), but questions of identifying and compensating for the dead time have also been studied in control theory [6], signal processing [7], medical imaging [8], mass spectrometry [9] and more.

The contribution of this paper is a limit theorem for the detection count distribution R' in a type II counter. The technique relies on providing formulas for the first and second moments of the waiting time between consecutive detections (also called the inter arrival time) in terms of the waiting time distribution of the original sequence of events and the dead time distribution, and on a limiting result for renewal processes, which will give an approximation formula for the average count rate as well as a Central Limit Theorem (CLT) approximation of the count distribution. Clearly, these approximations will be effective if the count rate is sufficiently high (with respect to the measurement time). In the renewal process literature, such approximations are referred to as "diffusion limits", and their applicability region is referred to as the "diffusion scale". For all practical purposes, the diffusion scale is obtained if the number of counts is sufficiently high.

1.2. Aims and motivation

The foundations of the mathematical treatment to the dead time phenomenon were laid in a series of papers by Feller [10] (1948), Hammersley [11] (1953), Takacs [1] (1956) and Pyke [12](1957). Not surprisingly, the deep interest in the dead time phenomenon came right after the rapid emergence of nuclear engineering, since radioactive measurements are often characterized by very high count rates, while the radiation detectors suffer from a non reducible dead time, caused by the physics of particle detection. These early works have presented seminal results in terms of the mathematical formalism and the existence of limit distributions, allowing us to refer to issues as the detector availability and average count losses. But often, from a practical point of view, numeric results are restricted to simplifying assumptions, such as exponential inter-arrival time, constant dead time and more. It should be mentioned that in the perspective of the 50's of the previous century, the assumption of an exponential inter-arrival time (which resolves to a Poisson distribution of the counts) seems to be fair, since in basic radioactive measurements, caused by simple radioactive decay, the inter-arrival time is indeed exponential.

As time went by, further and more elaborated examples of dead time effect emerged, where in many cases we can no longer assume an exponential inter-arrival waiting time; some still in the context of nuclear engineering, such as reactor noise experiments [13] and neutron multiplicity counting [14], but examples are also found in aerosol distribution and atmospheric sciences [15] (A list of dead time problems with non-Poisson count distribution is presented in Ref. [16]).

Clearly, the state of the art did not stay still since the 1950's. But the vast majority of later studies on the dead time effect are very narrow by nature, treating the dead time in a very practical aspect and in the context of a very specific process (often suggesting experimental or phenomenological corrections) and the theoretical problem, as described in the early papers, is rarely met. A recent literature review is provided in Ref. [17].

In the present study, we return to a more classic approach, offering rigorous mathematical analysis of a very general setting, but with a clear practical motivation. On one hand, we analyze what is perhaps the most general problem: we consider a general inter-arrival time and a general dead time distribution, and on the other hand, we will give practical examples for the theoretical model. As such, the main contribution of paper will be divided into two parts: in sections 2 and 3 we will discuss the general problem and provide the general formula, and in section 4 we will give examples of physical problems that are solved by the presented formulas.

2. Preliminaries and scientific background

2.1. Counting and renewal processes

The object in our study is associated with two sequences of random variables $\mathscr{L} = \{t_1, t_2, \ldots, t_n, \ldots\}$ and $\tau = \{\tau_1, \tau_2, \ldots, \tau_n, \ldots\}$ with the following properties:

1. $\{t_n\}_{n=1}^{\infty}$ is monotonically increasing.
2. Denoting $\theta_n = t_{n+1} - t_n$, $\{\theta_n\}_{n=1}^{\infty}$ are (positive) IID random variables, possessing first and second moments.
3. $\{\tau_n\}_{n=1}^{\infty}$ are non-negative IID random variables, possessing first and second moments.

Thus t_n indicates the time of the nth detection (or, in a more general setting, the nth event), θ_n gives the waiting time between the nth and $(n+1)$ event and $\{\tau_n\}_{n=1}^{\infty}$ describes the dead time following the nth detection. The Cumulative Distribution Function (CDF) of θ_i will be denoted by $F_\theta(t)$ and the CDF of τ_i by $F_\tau(t)$.

Throughout the study we will use the indicator function $\mathbb{I}$ of an event A: $\mathbb{I}_A = 1$ if A occurs, and $\mathbb{I}_A = 0$ otherwise. We define the stochastic process:

$$R(t) = \sum_{n=1}^{\infty} \mathbb{I}_{\{t_n \le t\}}, \qquad t \ge 0, \tag{2}$$

Eq. (2) might seem hard to follow, but it is simply a counter of how many events have occur prior to t: the indicator function is either 0 or 1, and a detection contributes to the sum if and only if the indicator is 1- which means that thedetection is prior to t. Since $\{\theta_n\}_{n=1}^{\infty}$ are IID, the stochastic process $R(t)$ forms a renewal process. We will often refer to $R(t)$ as the *counter* associated with $\mathscr{L}$.

The *Renewal function*, or *average counter* is defined by $m(t) = E[R(t)]$, and satisfies the elementary renewal theorem,

$$\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{E[\theta_1]}. \tag{3}$$

An even stronger version is the *central limit theorem* for renewal process, stating the following (Th. 14.6 of [2]). Let '⇒' denote convergence in distribution, and $\mathscr{N}(0,1)$ denote the standard normal distribution.

[1] the symbol # indicates to the size of a (finite) set.

Theorem 1. *Let $R(t)$ be a renewal process corresponding to waiting times $\{\theta_i\}i = 1_\infty$. Denote $\mu = E[\theta_1]$ and $\sigma^2 = Var[\theta_1] < \infty$. Then, as $t \to \infty$,*

$$\frac{R(t) - \frac{t}{\mu}}{\sigma\sqrt{t/\mu^3}} \Rightarrow \mathcal{N}(0,1).$$

The two aforementioned theorems provide first and second order approximations for the counting process $R(t)$ as the measurement time t gets large. We may state these approximations as follows. If $t \gg E[\theta]$ then

$$m(t) \approx \frac{t}{E[\theta_1]}; \quad R(t) \approx \mathcal{N}\left(\frac{1}{E[\theta_1]}t, \frac{Var[\theta_1]}{E[\theta_1]^3}t\right). \tag{4}$$

Note that these two approximations (often referred to as the diffusion scale approximation) require only the knowledge of the first and second moments of the waiting time. The first equation in (4) is fairly intuitive: it simply states that for large measurement times, the average number of detections is simple the measurement time divided by the average waiting time between consecutive counts. For the variance we do not know how to give a simple intuition.

2.2. Formal representation of detection counting

Recall that in a measurement system with extendable dead time, the recorded events form a rarefied sequence of $\mathcal{L}$, defined by equation (1). We denote the rarefied sequence by $\mathcal{L}' = \{t'_1, t'_2, \ldots, t'_n, \ldots\}$. The waiting times between consecutive events in the rarefied sequence are given by $\theta'_i = t'_{i+1} - t'_i$. It is not hard to see that if $\{\theta_i\}_{i=1}^\infty$ and $\{\tau_i\}_{i=1}^\infty$ are IID then $\{\theta'_i\}_{i=1}^\infty$ are IID as well. The counting process $R'(t)$, corresponding to $\{\theta'_i\}$ is thus also a renewal process. In order to use the result (3) and Theorem 1, so as to obtain a large time approximation for $R'(t)$, we need to compute the mean and variance of θ'_1. This is carried out in §3.

Since $\mathcal{L}'$ is a rarefaction of $\mathcal{L}$, there exists a unique monotone function $\phi : \mathbb{N} \to \mathbb{N}$ such that for all n, $t'_n = t_{\phi(n)}$ (in other words, $\phi(n)$ is the location of t'_n in the original time series). Let the *nth dead time chain length* be defined by

$$L_n = \max\left\{\ell \,\middle|\, \theta_{\phi(n)} < \tau_{\phi(n)}, \theta_{\phi(n)+1} < \tau_{\phi(n)+1}, \ldots, \theta_{\phi(n)+\ell-1} < \tau_{\phi(n)+\ell-1}\right\} = \phi(n+1) - \phi(n) - 1.$$

Then L_n is the number of lost detections between t'_n and t'_{n+1}. The waiting time between t'_n and t'_{n+1} is explicitly given by

$$\theta'_n = \sum_{\ell=0}^{L_n} \theta_{\phi(n)+\ell}. \tag{5}$$

Clearly, $\{\theta'_n\}$ is a series of IID random variables. The simple decomposition above of $\{\theta'_n\}$ in the rarefied series will prove useful when we compute the first two moments of θ'_1.

In what follows, we keep the notation $R'(t)$ for the renewal process corresponding to the waiting times $\{\theta'_n\}$, denote its renewal function by $m'(t)$. We refer to $R'(t)$ as the *Type II counter associated with $\{\theta_n\}$ under a dead time series τ.*

In our future analysis, we will always assume the that at time $t = 0$ there is a detection (or equivalently, that $t_1 = 0$). This will simplify the analysis, because other wise θ'_1 might have a different distribution than θ'_n $(n > 1)$. This does not effect the validity of the results, since at a high rate approximation, the first waiting time is negligible.

3. Large time approximation for type II counters

3.1. The first and second moments of θ'_1

The waiting time between the first and second detections is a random variable that depends on the length of the dead time chain, L_1, and on the first $L_1 + 1$ waiting times, as presented in equation (5). Specifically, the first accumulated waiting time is given by $\theta'_1 = \theta_1 + \theta_2 + \ldots + \theta_{L_1} + \theta_{L_1+1}$. It is convenient to write this as

$$\begin{aligned}\theta'_1 &= \sum_{n=0}^{\infty} \mathbb{1}_{\{L_1=n\}}(\theta_1 + \theta_2 + \ldots + \theta_n + \theta_{n+1}) \\ &= \sum_{n=0}^{\infty}\left(\mathbb{1}_{\{L_1=n\}}\sum_{j=1}^{n+1}\theta_j\right).\end{aligned} \tag{6}$$

Since $L_1 = n$ if and only if $\theta_j \le \tau_j$ for all $j = 1, 2, \ldots, n$ and $\theta_{n+1} > \tau_{n+1}$, we have

$$\mathbb{1}_{\{L_1=n\}} = \left(\prod_{j=1}^{n}\mathbb{1}_{\{\theta_j \le \tau_j\}}\right)\mathbb{1}_{\{\theta_{n+1} > \tau_{n+1}\}}$$

and thus

$$\begin{aligned}\theta'_1 &= \sum_{n=0}^{\infty}\left(\left(\prod_{j=1}^{n}\mathbb{1}_{\{\theta_j \le \tau_j\}}\right)\mathbb{1}_{\{\theta_{n+1} > \tau_{n+1}\}}\sum_{j=1}^{n+1}\theta_j\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{j=1}^{n}\mathbb{1}_{\{\theta_j \le \tau_j\}}\theta_j\left(\prod_{k=1,k\ne j}^{n}\mathbb{1}_{\{\theta_k \le \tau_k\}}\right)\mathbb{1}_{\{\theta_{n+1} > \tau_{n+1}\}} + \left(\prod_{k=1}^{n}\mathbb{1}_{\{\theta_k leq \tau_k\}}\right)\mathbb{1}_{\{\theta_{n+1} > \tau_{n+1}\}}\theta_{n+1}\right).\end{aligned} \tag{7}$$

We now fix the values of the random sequence $\tau = \{\tau_1, \tau_2, \ldots, \tau_n, \ldots\}$ and compute the conditional expectation $E[\theta'_1|\tau]$. By the independence of $\{\theta_i\}_{i=1}^\infty$, we have

$$E\left[\theta_1'\middle|\tau\right]=\sum_{n=0}^{\infty}\left(\sum_{j=1}^{n}E\left[\mathbb{1}_{\{\theta_j\le\tau_j\}}\theta_j\middle|\tau\right]\left(\prod_{k=1,k\ne j}^{n}P(\theta_k\le\tau_k|\tau)\right)P(\theta_{n+1}>\tau_{n+1}|\tau)\right.$$
$$\left.+\left[\prod_{k=1}^{n}P(\theta_k\le\tau_k|\tau)\right)E\left[\mathbb{1}_{\{\theta_{n+1}>\tau_{n+1}|\tau\}}\theta_{n+1}\middle|\tau\right]\right]. \tag{8}$$

Denote $P_{\tau_k}=P(\theta_k\le\tau_k|\tau)$. Then

$$\begin{aligned}P_{\tau_k}&=F_\theta(\tau_k)\\ 1-P_{\tau_k}&=1-F_\theta(\tau_k)\\ E\left[\mathbb{1}_{\{\theta_k\le\tau\}}\theta_k|\tau\right]&=\int_0^{\tau_k}xdF_\theta(x).\end{aligned} \tag{9}$$

and (3) can be written as:

$$E\left[\theta_1'\middle|\tau\right]=\sum_{n=1}^{\infty}\left[\sum_{j=1}^{n}\prod_{k=1,k\ne j}^{n}P_{\tau_k}(1-P_{\tau_{n+1}})\int_0^{\tau_k}xdF_\theta(x)+\prod_{k=1}^{n}P_{\tau_k}\int_{\tau_k}^{\infty}xdF_\theta(x)\right] \tag{10}$$

Since $\{\tau_k\}k=1_\infty$ are IID, we may apply the law of total expectation for each random value of τ_k. Denoting

$$S=E\left[P_{\tau_k}\right]=E[P_{\tau_1}]=\int_0^\infty\int_0^t dF_\theta(x)dF_\tau(t)$$

Computation of the second moment of θ_1' is fairly similar, but a bit more complicated. We first write

$$\left(\theta_1'\right)^2=\left[\sum_{n=0}^{\infty}\mathbb{1}_{\{L_1=n\}}\sum_{j=1}^{n}\theta_j+\theta_{n+1}\right]^2 \tag{12}$$

The events $L_1=n$ and $L_1=m$ are disjoint for $m\ne n$, and the product of the indicators in the first term in all off-diagonal terms is null. Hence

$$\begin{aligned}\left[\sum_{n=0}^{\infty}\left(\mathbb{1}_{\{L_1=n\}}\sum_{j=1}^{n}\theta_j+\theta_{n+1}\right)\right]^2&=\sum_{n=0}^{\infty}\mathbb{1}_{\{L_1=n\}}\left(\sum_{j=1}^{n}\theta_j+\theta_{n+1}\right)^2\\ &=\sum_{n=0}^{\infty}\mathbb{1}_{\{L_1=n\}}\left(\sum_{j=1}^{n}\theta_j\right)^2+2\sum_{n=0}^{\infty}\mathbb{1}_{\{L_1=n\}}\left(\sum_{j=1}^{n}\theta_j\right)\theta_{n+1}\\ &\quad+\sum_{n=0}^{\infty}\mathbb{1}_{\{L_1=n\}}(\theta_{n+1})^2\end{aligned}$$

The conditional expectation of the first term in (7), using the same argument as for the mean value, is

$$\begin{aligned}E\left[\theta_1'\right]&=\left(\sum_{n=0}^{\infty}nS^{n-1}(1-S)\right)\int_0^\infty\int_0^t xdF_\theta(x)dF_\tau(t)+\sum_{n=0}^{\infty}S^n\int_0^\infty\int_t^\infty xdF_\theta(x)dF_\tau(t)\\ &=(1-S)\frac{d}{dS}\left(\frac{1}{1-S}\right)\int_0^\infty\int_0^t xdF_\theta(x)dF_\tau(t)+\frac{1}{1-S}\int_0^\infty\int_0^t xdF_\theta(x)dF_\tau(t)\\ &=\frac{\int_0^\infty\int_0^t xdF_\theta(x)dF_\tau(t)}{1-S}+\frac{\int_0^\infty\int_0^t xdF_\theta(x)dF_\tau(t)}{1-S}=\frac{\int_0^\infty\int_0^\infty xdF_\theta(x)dF_\tau(t)}{1-S}\\ &=\frac{E[\theta_1]}{1-S}.\end{aligned} \tag{11}$$

$$E\left[\sum_{n=0}^{\infty}\mathbb{1}_{\{L_1=n\}}\left(\sum_{j=1}^{n}\theta_j\right)^2\middle|\tau\right]=E\left[\sum_{n=0}^{\infty}\mathbb{1}_{\{L_1=n\}}\sum_{j=1}^{n}\theta_j^2\middle|\tau\right]+E\left[\sum_{n=0}^{\infty}\mathbb{1}_{\{L_1=n\}}\sum_{j\neq k}^{n}\theta_j\theta_k\middle|\tau\right]=$$
$$\sum_{n=0}^{\infty}\sum_{j=1}^{n}\prod_{k=1,k\neq j}P_{\tau_k}^{n-1}(1-P_{\tau_{n+1}})E\left[\mathbb{1}_{\theta_j\le\tau_j}\theta_j^2\middle|\tau\right]+$$
$$\sum_{n=0}^{\infty}\sum_{j,k\le n,j\neq k}\prod_{l\le n,\ell\neq j,k}P_{\tau_\ell}(1-P_{\tau_{n+1}})E\left[\mathbb{1}_{\theta_k\le\tau_k}\theta_k|\tau\right]E\left[\mathbb{1}_{\theta_j\le\tau_j}\theta_j\middle|\tau\right]$$

and by averaging over τ we have:

$$E\left[\sum_{n=0}^{\infty}\mathbb{1}_{\{L_1=n\}}\left(\sum_{j=1}^{n}\theta_j\right)^2\right]=\sum_{n=0}^{\infty}nS^{n-1}(1-S)\int_0^{\infty}\int_0^{t}x^2dF_\theta(x)dF_\tau(t)+$$
$$\sum_{n=0}^{\infty}n(n-1)S^{n-2}(1-S)\int_0^{\infty}\left(\int_0^{t}xdF_\theta(x)\right)^2dF_\tau(t)=$$
$$\frac{1}{(1-S)}\int_0^{\infty}\int_0^{t}x^2dF_\theta(x)dF_\tau(t)+\frac{1}{(1-S)^2}\int_0^{\infty}\left(\int_0^{t}xdF_\theta(x)\right)^2dF_\tau(t) \tag{13}$$

The conditional expectation of the second term in (7) is:

$$E\left[\sum_{n=0}^{\infty}\mathbb{1}_{\{L_1=n\}}\left(\sum_{j=1}^{n}\theta_j\right)\theta_{n+1}\middle|\tau\right]=$$
$$E\left[\sum_{n=0}^{\infty}\left(\sum_{j=1}^{k}\left(\prod_{k=1,k\neq j}\mathbb{1}_{\{\theta_k\le\tau_t\}}\right)\mathbb{1}_{\{\theta_j\le\tau_j\}}\theta_j\right)\mathbb{1}_{\{\theta_{n+1}>\tau_{n+1}\}}\theta_{n+1}\middle|\tau\right]=$$
$$\sum_{n=0}^{\infty}\left(\sum_{j=1}^{k}\left(\prod_{k=1,k\neq j}E\left[\mathbb{1}_{\{\theta_k\le\tau_t\}}|\tau\right]\right)E\left[\mathbb{1}_{\{\theta_j\le\tau_j\}}\theta_j\middle|\tau\right]\right)E\left[\mathbb{1}_{\{\theta_{n+1}>u_{n+1}\}}\theta_{n+1}|\tau\right] \tag{14}$$

And thus:

$$E\left[\sum_{n=0}^{\infty}\mathbb{1}_{\{L_1=n\}}\left(\sum_{j=1}^{n}\theta_j\right)\theta_{n+1}\right]=$$
$$\sum_{n=0}^{\infty}nS^{n-1}\int_0^{\infty}\left(\int_0^{t}xdF_\theta(x)\right)dF_\tau(t)\int_0^{\infty}\left(\int_t^{\infty}xdF_\theta(x)\right)dF_\tau(t)= \tag{15}$$
$$\frac{1}{(1-S)^2}\int_0^{\infty}\left(\int_0^{t}xdF_\theta(x)\right)dF_\tau(t)\int_0^{\infty}\left(\int_t^{\infty}xdF_\theta(x)\right)dF_\tau(t)$$

Before we continue, we offer the following manipulation of equation (11), which will prove very useful later on. By equation (6) we can write

$$E^2[\theta_1]=\frac{1}{(1-S)^2}\left(\int_0^{\infty}xdF_\theta(x)\right)^2=\frac{1}{(1-S)^2}\int_0^{\infty}\left(\int_0^{\infty}xdF_\theta(x)\right)^2dF_\tau(t)=\frac{1}{(1-S)^2}\int_0^{\infty}\left(\int_0^{t}xdF_\theta(x)+\int_t^{\infty}xdF_\theta(x)\right)^2dF_\tau(t)=$$
$$\frac{1}{(1-S)^2}\left(\int_0^{\infty}\left(\int_0^{t}xdF_\theta(x)\right)^2dF_\tau(t)+\int_0^{\infty}\left(\int_t^{\infty}xdF_\theta(x)\right)^2dF_\tau(t)+2\int_0^{\infty}\left(\int_0^{t}xdF_\theta(x)\right)dF_\tau(t)\int_0^{\infty}\left(\int_t^{\infty}xdF_\theta(x)\right)dF_\tau(t)\right)$$

Hence

$$2E\left[\sum_{n=0}^{\infty}\mathbb{I}_{\{L_1=n\}}\left(\sum_{j=1}^{n}\theta_j\right)\theta_{n+1}\right]=$$
$$E^2[\theta_1]-\frac{1}{(1-S)^2}\left(\int_0^{\infty}\left(\int_0^{t}xdF_\theta(x)\right)^2dF_\tau(t)+\int_0^{\infty}\left(\int_t^{\infty}xdF_\theta(x)\right)^2dF_\tau(t)\right) \tag{16}$$

Finally, we carry out the exact same analysis on the last term in (7). The conditional expectation reads

$$E\left[\mathbb{I}_{\{L_1=n\}}\theta_{n+1}^2\middle|\tau\right]=\sum_{n=0}^{\infty}E\left[\prod_{i=1}^{n}\mathbb{I}_{\theta_i\le\tau_i}\middle|\tau\right]E\left[\mathbb{I}_{\theta_{n+1}>\tau_{n+1}}\theta_{n+1}^2\middle|\tau\right]$$
$$=\sum_{n=1}^{\infty}\prod_{i=1}^{n}P_{\tau_i}\int_{\tau_i}^{\infty}x^2f_\theta(x)dx$$

and following the averaging with respect to τ, we have:

$$E\left[\mathbb{I}_{\{L_1=n\}}\theta_{n+1}^2\right]=\frac{1}{1-S}\int_0^{\infty}\int_t^{\infty}x^2dF_\theta(x)dF_\tau(t) \tag{17}$$

Substituting the results of (9), (12) and (13) in (7) gives:

$$Var\left[\theta_1'\right]=\frac{E\left[\theta_1^2\right]}{1-S}+\frac{\int_0^{\infty}\left[\int_0^{t}xdF_\theta(x)\right]^2dF_\tau(t)}{(1-S)^2}-\frac{\int_0^{\infty}\left[\int_t^{\infty}xdF_\theta(x)\right]^2dF_u(t)}{(1-S)^2} \tag{18}$$

Equations (14) and (6), from a theoretical point of view, form the main contribution of the study. In these two, the mean value and variance of the inter-arrival time is given (in a closed form) in terms of the two relevant distributions: the original inter-arrival time and the dead time. Softer versions of these equations may be found in literature we have discussed earlier, considering specific distributions for either θ_i or τ_i (most of the time, exponential). One noticeable reduction of equations (14) and (6) is when the dead time is fixed, and

$$F_\tau(t)=\begin{cases}0; & t<\tau_0\\ 1; & t\ge\tau_0\end{cases}$$

in such case, we have:

$$E\left[\theta_1'\right]=\frac{\mathbb{E}[\theta_1]}{1-P_{\tau_0}} \tag{19}$$

$$Var\left[\theta_1'\right]=\frac{E\left[\theta_1^2\right]}{1-P_{\tau_0}}+\left[\frac{\int_0^{\tau_0}xdF_\theta(x)}{1-P_{\tau_0}}\right]^2-\left[\frac{\int_{\tau_0}^{\infty}xdF_\theta(x)}{1-P_{\tau_0}}\right]^2 \tag{20}$$

Where P_{τ_0} is simply the probability of an inter-arrival time less than the dead time τ_0.

3.2. Diffusion scale approximations for the count distribution

Once the first and second moments of θ_1' are computed, we obtain the following limit theorems for the Type II counters and the associated renewal function.

Theorem 1. *Let $m(t)$ be the renewal function associated with the waiting times $\{\theta_j\}_{j=1}^{\infty}$, and let $m'(t)$ be the renewal function of the Type II counter associated with $\{\theta_j\}_{j=1}^{\infty}$ under a dead time series τ. Then*

$$\lim_{t\to\infty}\frac{m'(t)}{t}=(1-S)\lim_{t\to\infty}\frac{m(t)}{t}.$$

Where $S=\int_0^{\infty}\int_0^{t}dF_\theta(x)dF_\tau(t)$ (*$F_\theta(x)$ and $F_\tau(x)$ as defined earlier).*

Theorem 2. *Let $\{\theta_j\}_{j=1}^{\infty}$ be IID and let $R'(t)$ be the corresponding Type II counter under dead time τ. Then, as $t\to\infty$,*

$$\frac{R'(t)-\frac{t}{\mu'(\tau)}}{\sigma'(\tau)\mu'(\tau)^{-3/2}\sqrt{t}}\Rightarrow\mathcal{N}(0,1),$$

where

$$\mu'(\tau)=\frac{\mathbb{E}[\theta_1]}{1-S}\quad\left(\sigma'(\tau)\right)^2=\frac{E\left[\theta_1^2\right]}{1-S}+\frac{\int_0^{\infty}\left[\int_0^{t}xdF_\theta(x)\right]^2dF_\tau(t)}{(1-S)^2}-\frac{\int_0^{\infty}\left[\int_t^{\infty}xdF_\theta(x)\right]^2dF_\tau(t)}{(1-S)^2}$$

Alternatively, one may state Theorems 1 and 2 as results providing approximations. That is, if $t\gg E[\theta_1]$ then

$$E\left[R'(t)\right]\approx\frac{1-P_\tau}{E[\theta_1]}t,\qquad R'(t)\approx\mathcal{N}\left(\frac{t}{\mu'(\tau)},\frac{(\sigma'(\tau))^2}{(\mu'(\tau))^3}t\right).$$

4. Examples and applications

So far, the study has focused on theoretical results, summed up in formulas (14) and (6) and the corresponding limit theorems. If we assume that the inter arrival time is exponential, then the results are well known, and previously presented in Refs. [1,11]. For the results to be significant from a practical point of view, we must consider examples of high rate counting procedures (with respect to the detector dead time) in which the inter-arrival times are non-exponential but IID.

First, we will start with a fairly simple example, comparing between exponential and uniform waiting times. The uniform waiting time does not have any deep scientific or technological significance, but they will demonstrate how the formulas are implemented in a non-trivial manner.

Then, in what follows, we give several examples of relevant counting processes: the first involve serial connection of counters, followed by two examples where the deviation from exponential waiting time is given through the so called pair correlation function (PCF). Two of the examples are directly connected to nuclear engineering. The third example, which comes from aerosol science, is included due the its relevance to this study.

In all examples, we will use the Probability Density Functions (PDF) $f_\theta(x)$ and $f_\tau(x)$ (rather than the CDF), using the equivalence

$$dF_\theta(x)=f_\theta(x)dx,\quad dF_\tau(x)=f_\tau(x)dx$$

4.1. Exponential vs. uniform waiting time distribution

As a first example, we compare through both simulation and equalities (14) and (6) between two simple examples.

In the first we consider an exponential waiting time, defined by the density function $f_\theta(x) = \lambda e^{-\lambda x}$. In the second, we take a waiting time uniformly distributed in the interval $[0,1]$, defined by the distribution

$$f_\theta(x) = \begin{cases} 1; & 0 \le x \le 1 \\ 0; & \text{else} \end{cases}$$

for both examples, we have computed the integral in (14) and (6), and verified the results with numeric simulation. A description of the simulation setting is as follows. A simulation of 5000 [sec] measurement was executed. To obtain a sample mean and a sample standard deviation, we have repeated the simulation 10^4 times. To measure the effect of the dead time, we have artificially inflicted dead time on the simulation, by deleting events that occur less then τ seconds after the previous event. This was done for 19 values of τ, evenly separated between $\tau = 0.05$ and $\tau = 0.95[sec]$.

Since we assume fixed dead time, we use the simplified formulas in (15) and (16). For the exponential waiting time, through direct calculations we have that:

$$\mu'(\tau) = \frac{e^{\lambda\tau}}{\lambda} \quad \left(\sigma'(\tau)\right)^2 = \frac{2e^{\lambda\tau}}{\lambda^2} + \left(\frac{\lambda\tau+1}{\lambda}\right)^2 - \left(\frac{e^{\lambda\tau}}{\lambda} - \frac{\lambda\tau+1}{\lambda}\right)^2$$

And for the uniform distribution, we have computed:

$$\mu'(\tau) = \frac{1}{2(1-\tau)}; \quad \left(\sigma'(\tau)\right)^2 = \left[\frac{1}{3(1-\tau)} + \frac{\tau^4}{4(1-\tau)^2} - \frac{(1-\tau^2)^2}{4(1-\tau)^2}\right].$$

To insure that both associated renewal process will have the same reaction rates (with zero dead time), we take $\lambda = 2$ for the numeric simulations.

Fig. 1 shows a histogram of the simulated values for the Counts Per Second (CPS) in all simulations, with a dead time of $\tau = 0.2[sec]$. Running a chi-squared test on the values accepted the null hypothesis with a p-value of 0.479.

Figs. 2 and 3 shows the sample mean of the CPS as a function of τ, (for both simulations) together with the analytic prediction. First, we can see there is an excellent correspondence between the simulation and analytic prediction. Next, we see that the dead time effect has a very different nature. In the uniform waiting time, the count reduction is precisely linear, up until a total loss of counts at $\tau = 1$. In the exponential model, the count reduction also follow an exponential law, and the counts can vanish only at $\tau \to \infty$.

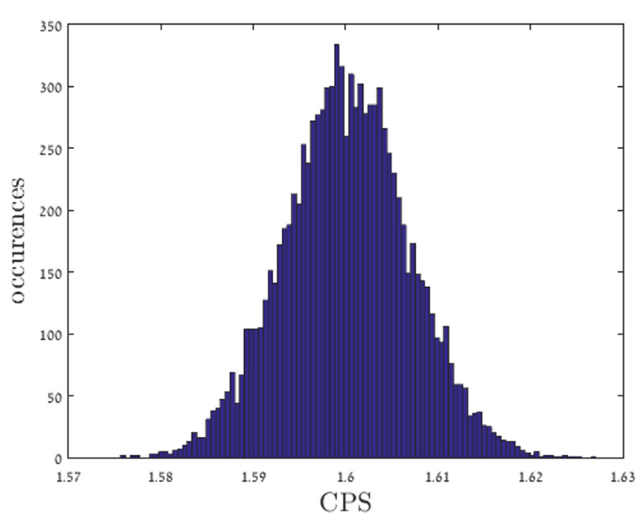

Fig. 1. Sampled CPS histogram, $\tau = 0.2[sec]$

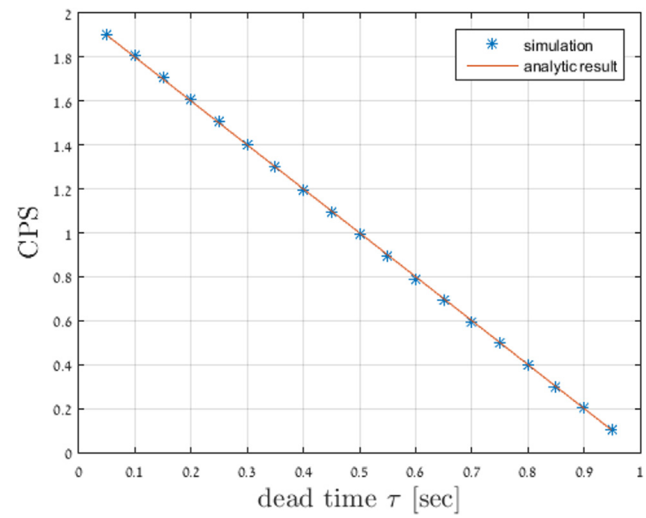

Fig. 2. Sampled mean of the CPS with a uniformly distributed waiting time.

Figs. 4 and 5 show the sampled standard deviation for both examples (again, as a function of τ), and now we see an even more distinctive behavior. For the exponential model, the reduction in not exactly exponential, since the exponent is multiplied by a linear term, but the behavior is still very close to exponential. In the uniform distribution, on the other hand, the standard deviation is a rational function (as a function of τ), that once again vanishes at $\tau = 1$. It is surprising to find out that in the uniform distribution, the variance of the count distribution is not a monotonic function of τ, but has a local minimum and maximum (see Fig. 4).

As we have mentioned, the uniform distribution does not carry any scientific of engineering significance (that the authors are aware of), but it does serve as a very elegant example for demonstrating the strong dependence between the dead time effect to the original waiting time.

4.2. Serially connected components

In most counting models, the model consists of two elements:

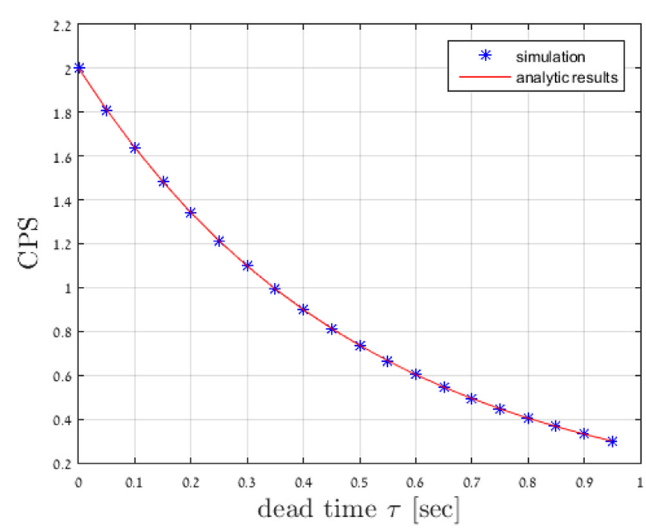

Fig. 3. Sampled mean of the CPS with exponentially distributed waiting time.

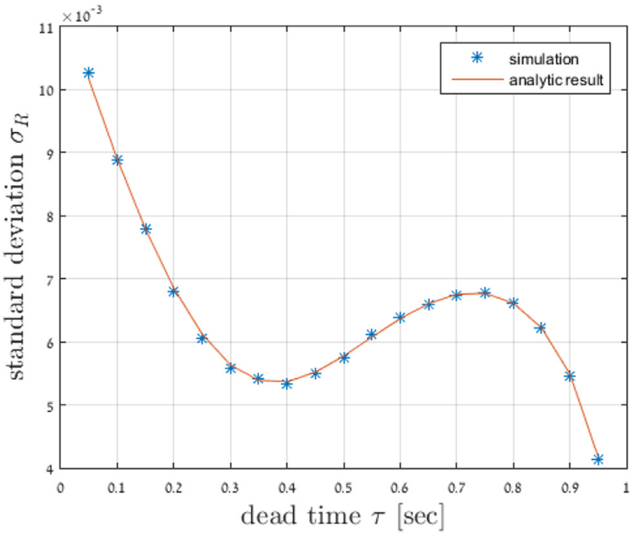

Fig. 4. Sampled standard deviation of the CPS in a uniformlly distributed waiting time.

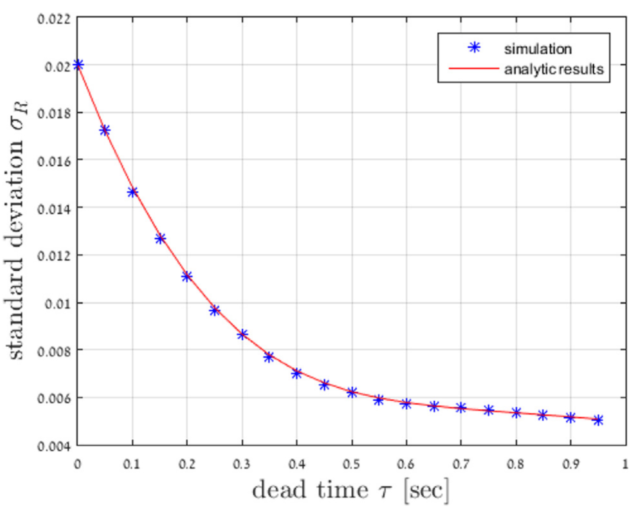

Fig. 5. Sampled standard deviation of the CPS in an exponentially distributed waiting time.

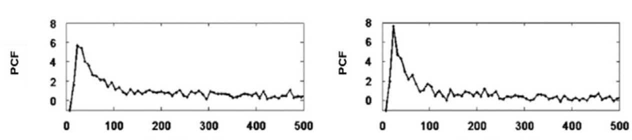

Fig. 6. Sampled PCF as a function of the lag time. Figures taken from Larsen [15].

the source and the detector (counter). However, in real life, the detector itself is often constructed as a serial connection of two (or more) elements [17]. In radiation measurements, for instance, the detector module is often constructed from the physical detector, which is a tube containing materials that create a noticeable reaction with the detected particles, serially connected to a signal amplifier, or a computer sampling card. This is not a unique attribute of radiation measurements, and often counters consist of two serially connected elements: a physical component (A) which detects the events, and a second component (B), which digitally records them. In the terminology used so far, if both components (A) and (B) have a dead time period (and, as mentioned, to some extent they always do), then the rarefied (output) signal from component (A) becomes the non-rarefied input signal for component (B).

To demonstrate the applicability of our results, we will consider the following two settings: in both we assume that the original inter-arrival time is exponentially distributed with an average waiting time of $1/\lambda$. In the first setting we assume that component (A) has a fixed non extendable (Type I) dead time τ_0 and component (B) has a fixed extendable (Type II) dead time τ_1. In the second we will look at a slightly more elaborate setting, where the duration of the dead time of component (B) has an exponential distribution, with a mean value τ_1.

4.2.1. Example I: both components suffer from fixed dead time

In the first example, since the original waiting time between consecutive events has an exponential distribution, the waiting time between the recuperation of the detector (A) and the following detection is once again exponential with the exact same parameters. Therefore, the inter-arrival time in the rarefied series created by detector (A)- which serves as the unrarefied series for component (B) - is of the form $\theta_i = x_i + \tau_0$, where x_i is a random variable exponentially distributed with mean value $1/\lambda$. The PDF of $\{\theta_j\}_{j=1}^{\infty}$ is given by $f_\theta(x) = U_0(x-\tau_0)\lambda e^{-\lambda(x-\tau_0)}$ (where $U_0(x) = 0$ if $x < 0$ and $U_0(x) = 1$ if $x \geq 0$), and:

$$E[\theta_1] = E[x_1] + \tau_0 = \frac{1}{\lambda} + \tau_0$$

$$Var[\theta_1] = Var[x_1] = \frac{1}{\lambda^2}$$

$$E\left[\theta_1^2\right] = Var[\theta_1] + E^2[\theta_1] = \frac{1}{\lambda^2} + \left(\frac{1}{\lambda} + \tau_0\right)^2$$

Again, since we assume that the dead time of component (B) is fixed, we may apply formulas (15) and (16). We now divide into two cases: $\tau_1 \leq \tau_0$ and $\tau_1 > \tau_0$. For the first, the dead time of the second component (B) is irrelevant, since the minimal waiting time in the signal entering the component is τ_0. For $\tau_1 > \tau_0$ we compute directly:

$$P_{\tau_1} = 1 - e^{-\lambda(\tau_1-\tau_0)}, \qquad \int_0^{\tau_1} \theta f_\theta(\theta)d\theta = \lambda \int_{\tau_0}^{\tau_1} \theta e^{-\lambda(\theta-\tau_0)}$$

$$d\theta = \left(\frac{1}{\lambda} + \tau_0\right) - \left(\frac{1}{\lambda} + \tau_1\right)e^{-\lambda(\tau_1-\tau_0)}$$

$$\int_{\tau_1}^{\infty} \theta f_\theta(\theta)d\theta = \left(\frac{1}{\lambda} + \tau_1\right)e^{-\lambda(\tau_1-\tau_0)}$$

and we have:

$$\mu = E\left[\theta_1'\right] = e^{\lambda(\tau_1-\tau_0)}\left(\frac{1}{\lambda} + \tau_0\right);$$

$$\sigma^2 = Var\left[\theta_1'\right] = e^{\lambda(\tau_1-\tau_0)}\left(\frac{1}{\lambda^2} + \left(\frac{1}{\lambda} + \tau_0\right)^2\right) +$$

$$\left(\frac{\left(\frac{1}{\lambda} + \tau_0\right) - \left(\frac{1}{\lambda} + \tau_1\right)e^{-\lambda(\tau_1-\tau_0)}}{1 - e^{-\lambda(\tau_1-\tau_0)}}\right)^2 - \left(\frac{\left(\frac{1}{\lambda} + \tau_1\right)e^{-\lambda(\tau_1-\tau_0)}}{1 - e^{-\lambda(\tau_1-\tau_0)}}\right)^2$$

4.2.2. Example II: component (A) suffer from fixed dead time, and component (B) has an exponentially distributed dead time

In the second example, we use the set of equations (6) and (14), with:

$$f_\theta(x) = U_0(t-\tau_0)\lambda e^{-\lambda(t-\tau_0)};$$

$$f_\tau(x) = \frac{1}{\tau_1} e^{\frac{1}{\tau_1}x}$$

Through direct computation

$$S = \int_0^\infty \left(\int_0^t f_\theta(x)dx\right) f_\tau(t)dt = e^{-\frac{\tau_0}{\tau_1}}\left(1 - \frac{1}{1+\lambda\tau_1}\right)$$

and thus

$$\mu(\tau_0, \tau_1) = \frac{\frac{1}{\lambda} + \tau 0}{1 - e^{-\frac{\tau_0}{\tau_1}}\left(1 - \frac{1}{1+\lambda\tau_1}\right)}$$

For the variance, computations are more complicated. First, we have that $E[\theta_1^2] = \frac{1}{\lambda^2} + \left(\frac{1}{\lambda} + \tau_0\right)^2$. For the integral terms in (?) we have:

$$\int_0^\infty \left[\int_0^t x f_\theta(x)\right]^2 f_\tau(t)dt = \int_{\tau_0}^\infty \left[\int_{\tau_0}^t x\lambda e^{-\lambda(x-\tau_0)}dx\right]^2 \frac{1}{\tau_1} e^{-\frac{t}{\tau_1}}dt$$

$$\int_0^\infty \left[\int_t^\infty x f_\theta(x)\right]^2 f_\tau(t)dt = \int_0^{\tau_0} \left[\int_{\tau_0}^\infty x\lambda e^{-\lambda(x-\tau_0)}dx\right]^2 \frac{1}{\tau_1} e^{-\frac{t}{\tau_1}}dt + \int_{\tau_0}^\infty \left[\int_t^\infty x\lambda e^{-\lambda(x-\tau_0)}dx\right]^2 \frac{1}{\tau_1} e^{-\frac{t}{\tau_1}}dt$$

The last integrations can be executed fully (although the results would be fairly lengthy), and once substituted in (6) and (14) would then give explicit formulas for σ^2, from which the parameters of the diffusion scale approximation can be computed.

4.3. Pair correlation function

The exponential distribution with parameter λ is defined by the its PDF $F_x(t) = P(x \le t) = (1 - e^{-\lambda t})$. Renewal processes with exponential inter-arrival times are of utmost importance, since they describe a memory-less waiting time between consecutive events. An alternative definition of exponential inter-arrival time is the following: if we denote by $P(t, \Delta t|t_0)$ the probability of an event in the interval $[t_0+t, t_0+t+\Delta]$ *given an event at* t_0, then $P(t, \Delta t|t_0)$ satisfies:

$$P(t, \Delta t|t_0) = \lambda\Delta t + o(\Delta t) \tag{21}$$

A function $\eta(t)$ is called the Pair Coefficient Function (PCF) if (1) can be replaced by

$$P(t, \Delta t|t_0) = \lambda(1+\eta(t))\Delta t + o(\Delta t) \tag{22}$$

Since the left hand is a probability function, it is assured that $\eta(t) \ge -1$. The PCF can be interpreted as a measure of the deviation from exponential distribution where $\eta(t) < 0$ means a negative correlation between an event at $t = t_0$ and an event in the interval $[t_0+t, t_0+t+\Delta]$ and $\eta(t) > 0$ means a positive correlation between an event at $t = t_0$ and an event in the interval $[t_0+t, t_0+t+\Delta]$ (and if $\eta = 0$, we have a memory-less property, as in (1)). Any counting process where the inter-arrival time satisfies (2) must be a renewal process, since the right hand side only depends on t (and not t_0). In the following section, we give two physical examples of a counting procedure with a non negligible dead time, where the PCF following a detection is not zero.

Before we continue, we notice that the previous example can also be formulated by the PCF, with

$$\eta = \begin{cases} -1; & t < \tau_0 \\ 0; & t \ge \tau_0 \end{cases}$$

4.31. Neutron multiplicity counting

Radioactive decay has an exponentially distributed waiting time between consecutive nuclear events. If in each nuclear reaction a single particle is emitted, then the waiting time between consecutive detections is once again exponential. However, if in each reaction a number of particles are emitted (for example, is spontaneous fissions of ^{240}Pu or ^{252}Cf) then the probability of a detection at an infinitesimal interval dt starting t seconds after a detection at t_0 is give by Ref. [14]:

$$P(t, dt|t_0) = \left(C_0 + C_1 e^{-\lambda t}\right)dt = C_0\left(1 + \frac{C_1}{C_0}e^{-\lambda t}\right)dt \tag{23}$$

The coefficient C_0 describes the amplitude of the uncorrelated source ("accidentals") while C_1 describes the detection rate of the neutrons that are correlated with the detection at $t = t_0$, which decays exponentially with a coefficient λ. Equation (3) is in wide use, and is the basic consideration in the theory of neutron coincidence counting, aimed to determine the multiplication and mass of a sample by measuring the so called *doubles to singles* rate [18]. Clearly, equation (3) is a special case of (2), with $\eta(t) = \frac{C_1}{C_0}e^{-\lambda t}$.

The dead time effect in neutron multiplicity counting is a well studied topic, due to its applicable nature. The effect on the first and second moments of the count distribution in coincidence counting was studied in Ref. [14], using some analytic considerations, but the study is based on a phenomenological model, and empirical fitted data. In Ref. [19], an applicable closed form of the formulas in Ref. [14] was introduced. The dead time effect on higher moments was modeled in Ref. [20] using multi dimensional distribution (and later extended in Ref. [21]). A more pragmatic approach for estimating the dead time parameter was given in Ref. [22], and the list goes on.

To implement the formulas presented in this study, we must first derive the waiting time distribution from (3). We denote by $F(t)$ the CDF of the inter-arrival time. For simplicity, we assume a detection at $t = 0$. The probability for a first event in the interval $[t, t+dt]$ following a detection at $t = 0$ is given by:

$$F(t+dt) - F(dt) = (1 - F(t))\left(C_0 + C_1 e^{-\lambda t}\right)dt$$

The right hand side of the above is a product of the probabilities of two independent events: the first is that there are no detections in the interval $[0, t]$ and the second is a detection in the interval $[t, t+dt]$. By taking the limit $dt \to 0$, we obtain the equation for $F(t)$

$$\frac{dF}{dt} = (1 - F(t))\left(C_0 + C_1 e^{-\lambda t}\right)$$

admitting a solution:

$$F(t) = 1 - Exp\left[C_0 t + \frac{C_1}{\lambda}\left(1 - e^{-\lambda t}\right)\right] \tag{24}$$

(in the last, we also account for the initial condition $F(0) = 0$). Finally, the PDF of the waiting time is given by:

$$f_\theta(t) = \frac{dF}{dt} = Exp\left[C_0 t + \frac{C_1}{\lambda}\left(1 - e^{-\lambda t}\right)\right]\left(C_0 + C_1 e^{-\lambda t}\right)$$

For the variance, explicit integration is non trivial, due to the factor $tf_\theta(t)$ appearing under the integration- but from a practical point of view, once the parameters are known, numeric integration is clearly doable.

For the average count rate, the correction is trivial: for a dead time τ, the dead time losses are given by multiplying the theoretical count rate by $1 - E[F(\tau)] = E\left[Exp\left[C_0\tau + \frac{C_1}{\lambda}(1 - e^{-\lambda\tau})\right]\right]$. In Ref. [14], the correction factor is given by $Exp[C_0\tau_0 + C_1\tau_0]$, which is obtained by a first order approximation of the inner exponent in the earlier expression, under the assumption that the dead time is fixed.

4.3.2. Spatial distribution of aerosol particles

Measuring the spatial distribution of aerosol particles is a basic topic in atmospheric and aerosol science. Often, it is assumed that the aerosol particles follow a Poisson spatial distribution [23], which means that if the space is scannedin a constant rate, an exponential waiting time between detections will be measured. However, the Poisson distribution assumption is still in debate, and in Ref. [15] we find a discussion on the subject. In particular, experimental results for the measured PCF as a function of t (referred to as the lag time) were presented, showing explicit non exponential behavior. Figure (6) below is taken from Ref. [15], showing the sampled PCF as a function of the lag time (for two different data sets, and for particle larger that $1\mu m$).

The results are exactly consistent with the situation analyzed in the present study: the sharp drop on the left hand side of both plots indicate a detector dead time of approximately $\tau_0 = 30\mu s$, and the strict positive values in the interval $u_0 \le t \le 150[\mu Sec]$ (approximately) clearly indicate non exponential “clustering”. This was already observed and discussed in Ref. [16], but the analysis was not complete. Using the results in the present study, an explicit analytic correction (for large scale measurements) can be given as a function of η. First, as in section 1, the CDF is defined by the ODE:

$$\frac{dF}{dt} = (1 - F(t))\lambda(1 + \eta(t)); F(0) = 0;$$

Admitting the solution

$$F(t) = 1 - Exp\left[\int_0^t \lambda(1 + \eta(x))dx\right] \quad (25)$$

and the PDF of the waiting time is give by:

$$f_\theta(t) = \frac{dF}{dt} = Exp\left[\int_0^t \lambda(1 + \eta(x))dx\right]\lambda(1 + \eta(t))$$

To obtain a full correction term, η must be known, and clearly, we cannot say anything about η. But once η is known, the correction is straight forward, and explicitly given.

5. Concluding remarks

This paper studies a limiting distribution of detection counts in a Type II (extendable dead time) counter, under a general waiting time distribution between consecutive events (inter-arrival time) and a randomly distributed dead time. In particular, explicit formulas for the distribution are provided in terms of the CDF's of the inter-arrival time distribution and the dead time distribution.

Following the theoretical results, we have introduced three examples for applications: serially connected components, neutron multiplicity counting and spatial distribution of aerosol particles. In all three examples, the deviation from a Poisson distribution of the number of counts can be measured by the pair coefficient function.

Future work would include the case of non-extendable dead time (Type I counters). The main challenge in this case is that the time period between the recuperation of the counter and the previous event is not described simply by the dead time distribution, and the method we have used in this paper fails.

Acknowledgment

Fig. 1 was taken from Ref. [15], under approval of Elsevier, order No. 4475891403744.

Appendix A. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.net.2019.04.015.

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