A DURRETT-REMNENIK PARTICLE SYSTEM IN $\mathbb{R}^d$

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This paper studies a branching-selection model of motionless particles in $\mathbb{R}^d$, with nonlocal branching, introduced by Durrett and Remenik in dimension 1. The assumptions on the fitness function, $F$, and on the inhomogeneous branching distribution, are mild. The evolution equation for the macroscopic density is given by an integro-differential free boundary problem in $\mathbb{R}^d$, in which the free boundary represents the least $F$-value in the population. The main result is the characterization of the limit in probability of the empirical measure process in terms of the unique solution to this free boundary problem.

1. Introduction. Consider a system of motionless particles living in $\mathbb{R}^d$, $d \geq 1$, that undergo branching and selection. The branching occurs nonlocally, where a particle at location $y$ gives birth at rate 1 to a new particle at a location distributed according to a probability density $\rho(y, \cdot)$. Upon each branching, the particle with least $F$-value among the living particles, including the newborn, is removed. Thus the number of particles, denoted $N$, remains constant. Here, $F : \mathbb{R}^d \to \mathbb{R}$ is a given fitness function. This model was introduced and studied in [13] for $d = 1$ and $F(x) = x$, and its hydrodynamic limit was characterized in terms of an integro-differential free boundary problem (FBP).

This result is extended in this paper. Under mild assumptions on $\rho$ and $F$, it is shown that the empirical measure process converges in probability to a deterministic path in measure space, whose density uniquely solves an integro-differential FBP in $\mathbb{R}^d$. This is the problem of finding a pair $(u, \ell)$, where $\ell : \mathbb{R}_+ \to \mathbb{R}$ is càdlàg and the function $u : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}_+$ is $C^{0,1}$ in $\{(x, t) \in \mathbb{R} \times \mathbb{R}_+ : F(x) > \ell_t\}$, and one has

$$\begin{align*}
\frac{\partial}{\partial t} u(x, t) &= \int_{\mathbb{R}^d} u(y, t) \rho(y, x) dy \quad (x, t) : F(x) > \ell_t, \ t > 0, \\
u(x, t) &= 0 \quad (x, t) : F(x) \leq \ell_t, \ t > 0, \\
\int_{\mathbb{R}^d} u(x, t) dx &= 1, \\
u(\cdot, 0) &= u_0, \ \ell_0 = \lambda_0.
\end{align*}$$

The free boundary $\ell$ represents the least $F$-value in the population, and $u$ is the macroscopic density.

Particle systems with selection have mostly been studied in dimension 1. They were proposed in [6, 7] as models for natural selection in population dynamics, where the position of a particle represents the degree of fitness of an individual to its environment. The monograph [8] studied systems of Brownian particles with selection and characterized their hydrodynamic limit via a FBP. Brownian particles with branching and selection, referred to as
$N$-particle Branching Brownian motion ($N$-BBM) were introduced in [17], and studied at the hydrodynamic limit in [4, 10]. Other works on hydrodynamic limits related to FBP include [11], where $N$-BBM with nonlocal branching was analyzed, and [9] where a variant of the symmetric simple exclusion process was studied. A branching-injection-selection model was studied in [1] via a weak FBP formulation.

In higher dimension, [5] studied the asymptotic shape of the cloud of particles of the $N$-BBM with fitness function $F(x) = \|x\|$ and $F(x) = \lambda \cdot x$. The papers [2, 3] considered the hydrodynamic limit of the $N$-BBM with $F(x) = -\|x\|$, characterizing it in terms of a FBP, studying convergence rates as well as long time behavior. This treatment used in a crucial way the symmetry of $F$, by which the radial projection of the macroscopic dynamics is governed by an autonomous equation, and thus the motion of the free boundary is dictated by an equation in one dimension.

Our motivation in this work is to study a branching-selection model (though, not the $N$-BBM) for which the macroscopic evolution equation is truly multidimensional in the sense that it does not reduce to a FBP in one spatial dimension.

In §2, the precise details of the model are introduced and the main result, Theorem 2.2, is stated. §3 contains the proof of Theorem 2.2. First, in §3.1, uniqueness of solutions to the FBP is proved. A sketch of the main idea behind this proof is given at the beginning of §3.1. It is based on the observation that, given two solutions $(u, \ell)$ and $(v, m)$, the $L^1$ distance between $u(\cdot, t)$ and $v(\cdot, t)$ can be estimated by the $L^1$ distance between their restrictions to $F^{-1}(\ell_t \vee m_t, \infty)$ (see (3.1)). Then, in §3.2, it is shown that the empirical measure processes form a tight sequence and that limits are supported on solutions to the FBP, which completes the proof. One of the tools that has been used in earlier work (e.g. [8]) to establishing hydrodynamic limits for particle systems with selection is the method of barriers. These are Trotter-type discrete approximations to the dynamics that constitute upper and lower bounds on any limit point of the empirical measure process and any solution to the FBP, by which it possible to show existence of a limit and uniqueness of solutions. The argument that we provide in §3.2 is also based on discrete approximations, however they are different from barriers in an important way, that is, they are constructed from a given solution. As a result these approximations alone do not imply uniqueness. Their goal, rather, is to prove that limits satisfy the integro-differential equation.

Notation. Denote $\mathbb{R}_+ = [0, \infty)$. Let $\mathcal{B}(\mathbb{R}^d)$ denote the class of Borel subsets of $\mathbb{R}^d$. In $\mathbb{R}^d$, denote the Euclidean norm by $\| \cdot \|$ and let $\mathcal{B}_r = \{ x \in \mathbb{R}^d : \|x\| \leq r \}$. Denote by $\mathcal{M}(\mathbb{R}^d)$ the space of finite signed Borel measures on $\mathbb{R}^d$ endowed with the topology of weak convergence. Let $\mathcal{P}(\mathbb{R}^d) \subset \mathcal{M}_+(\mathbb{R}^d) \subset \mathcal{M}(\mathbb{R}^d)$ denote the subsets of probability and, respectively positive measures, and give them the inherited topologies. For $\mu, \nu \in \mathcal{M}_+(\mathbb{R}^d)$, write $\mu \subseteq \nu$ if $\mu(A) \leq \nu(A)$ for all $A \in \mathcal{B}(\mathbb{R}^d)$.

For $(X, d_X)$ a Polish space let $C(\mathbb{R}_+, X)$ and $D(\mathbb{R}_+, X)$ denote the space of continuous and, respectively, càdlàg paths, endowed with the topology of uniform convergence on compacts and, respectively, the Skorohod $J_1$ topology. Let $C_0(\mathbb{R}_+, \mathbb{R}_+)$ denote the subset of $C(\mathbb{R}_+, \mathbb{R}_+)$ of nondecreasing functions that vanish at zero. For $f : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ denote

\[
\| f \|_T^\infty = \sup\{ \| f(s) \| : s \in [0, T] \},
\]

\[
\| f \|_T = \sup\{ \| f(s) \| : 0 \leq s \leq t \leq (s + \delta) \wedge T \},
\]

2. Model and main result. The particle system model is described as follows. An initial configuration is given by $\{ x^i, 1 \leq i \leq N \}$, where $x^i$ are $\mathbb{R}^d$-valued random variables. An independent exponential clock of rate 1 is attached to each particle, and when the clock of a particle rings, it gives birth to a new one. The location of a particle born from a particle at $y$
A solution to (2.1) is defined as a member of $X$ denoted by $(D_a, \xi)$. In a more general setting not covered in this paper, where the mass conservation condition is nondecreasing, (2.1) is drawn according to a probability density $\xi$ by living particles are labeled as 1 of the particle that gets removed is transferred to the new particle, so that at all times, the total order on the fitness function of the particle that has the least $\xi_i = \sum \delta_X(t)$, and the empirical measure by $\xi^N_t = N^{-1} \xi^N_i$. In particular, the initial empirical measure is $\xi^N_0 = N^{-1} \sum \delta_x$.

Following is our assumption on $F, \rho$ and $\xi^N_0$. Denote $\ell^N_0 = \min \{ F(y) : y \in \text{supp}(\xi^N_0) \}$.

**Assumption 2.1.** (i) $F \in C(\mathbb{R}^d, \mathbb{R})$, $\inf_x F(x) = -\infty$, $\sup_x F(x) = \infty$, and for every $a \in \mathbb{R}$, $F^{-1}(a)$ has Lebesgue measure zero.

(ii) There exists a probability density $\tilde{\rho}$ and a constant $\tilde{c}$ such that $\rho(x,y) \leq \tilde{c} \tilde{\rho}(y-x)$. Moreover, $\rho(x,y)$ is continuous in $y$ uniformly in $(x,y)$, and for $a \in \mathbb{R}$ and $x \in F^{-1}(a, \infty)$, one has $\int_{F^{-1}(a, \infty)} \rho(x,y) dy > 0$.

(iii) As $N \to \infty$, $(\xi^N_0, \ell^N_0) \to (\xi_0, \lambda_0)$ in probability, where the latter tuple is deterministic, $\xi_0(dx) = u_0(x)dx$ and $\lambda_0 \in \mathbb{R}$. Moreover, $u_0$ is bounded and continuous on $F^{-1}(\lambda_0, \infty)$ (and necessarily vanishes on $F^{-1}(-\infty, \lambda_0)$), and for every $\delta > 0$ and $\lambda \geq \lambda_0$, $\int_{F^{-1}(\lambda, \lambda+\delta)} u_0(x)dx > 0$.

Further notation is $\ell^N_t = \min \{ F(y) : y \in \text{supp}(\xi^N_t) \}$ for the minimal $F$-value of all living particles at time $t$, and $J^N_t$ for the removal counting process. By construction, $J^N_t$ is a Poisson process of rate $\lambda$.

Next is an equation for the macroscopic dynamics. Denote by $X$ the set of pairs $(u, \ell)$, where $\ell \in D(\mathbb{R}_+, \mathbb{R})$ and $u : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}_+$ is $C^{0,1}$ in $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ : $F(x) > \ell_t$ and bounded on $\mathbb{R}^d \times [0,T]$ for any $T$. Consider the system

$$
\begin{cases}
(i) & \partial_t u(x,t) = \int_{\mathbb{R}^d} u(y,t)\rho(y,x)dy & (x,t) : F(x) > \ell_t, t > 0, \\
(ii) & u(x,t) = 0 & (x,t) : F(x) \leq \ell_t, t > 0, \\
(iii) & \int_{\mathbb{R}^d} u(x,t)dx = 1, & t > 0, \\
(iv) & u(\cdot, 0) = u_0, \ell_0 = \lambda_0.
\end{cases}
$$

A solution to (2.1) is defined as a member of $X$ satisfying (2.1).

**Theorem 2.2.** Let Assumption 2.1 hold. Then there exists a unique solution to (2.1), denoted by $(u, \ell)$. Moreover, $\ell - \lambda_0 \in C^1(\mathbb{R}_+, \mathbb{R}_+)$. Furthermore, $(\xi^N_t, \ell^N_t) \to (\xi_t, \ell_t)$, in probability, in $D(\mathbb{R}_+, \mathcal{P}(\mathbb{R}^d)) \times D(\mathbb{R}_+, \mathbb{R}_+)$, where $\xi_t(dx) = u(x,t)dx$.

**Remark 2.3.** Because, as stated above, the $\ell$ component of any solution to (2.1) is nondecreasing, (2.1)(i) can be written in integral form as

$$
u(x,t) = u_0(x) + \int_0^t \int_{\mathbb{R}^d} u(y,s)\rho(y,x)dydt, \quad (x,t) : F(x) > \ell_t, t > 0.
$$

In a more general setting not covered in this paper, where the mass conservation condition is replaced by $\int_{\mathbb{R}^d} u(x,t)dx = m(t)$, with $m(t)$ given, $\ell$ need not be nondecreasing. In this case
the system (2.1) (with 1 replaced by \( m(t) \) on the r.h.s. of (iii)) is not sufficient for character-
ing \((u, \ell)\). Roughly speaking, a boundary condition \( u(\ell_t, t) = 0 \) should be added at times
when \( \ell_t \) is decreasing. A precise way to write this is
\[
u(x, t) = u_0(x)1_{\{\tau(t,x) = 0\}} + \int_0^t \int_{Z(x)} u(y, s)i(y, x)dydt, \quad (x, t) : F(x) > \ell_t, t > 0,\]
where \( \tau(x, t) = \inf\{s \in [0, t] : \ell_s < x \text{ for all } \theta \in (s, t)\} \).

**Remark 2.4.** A gap was found in the proof of uniqueness of solutions to the integro-
differential FBP in [13] (this is equation (FB), corresponding to (2.1) above in the case \( d = 1 \) and \( F(x) = x \)). See [14, Remark 2.10 added in 2023] for details. Our proof, which
addresses uniqueness via a different approach, validates the uniqueness statement of [13].

## 3. Proof of main result.

The proof of Theorem 2.2 proceeds in two main steps. In §3.1, it is shown that uniqueness holds for solutions to (2.1). In §3.2, it is shown that tightness
holds and that limits are supported on solutions to this equation.

### 3.1. Uniqueness.

For a quick sketch of the idea behind the proof of uniqueness, consider
the case \( d = 1 \) and \( F(x) = x \). If \((u, \ell)\) and \((v, m)\) are solutions then, for each \( t > 0 \), \( u(x, t) \)
vanishes for \( x < \ell_t \) and \( v(x, t) \) vanishes for \( x < m_t \). This and the fact that \( u \) and \( v \) have the
same mass imply
\[
\int_R |u(x, t) - v(x, t)|dx \leq 2 \int_{\ell_t \vee m_t} |u(x, t) - v(x, t)|dx.
\]
For both \( u \) and \( v \), the r.h.s. now involves only \((x, t)\) for which the integro-differential equation
(2.1)(i) holds. Integrating it over time allows the use of Gronwall’s lemma.

Before implementing this we need the following.

**Lemma 3.1.** Let \((u, \ell) \in \mathcal{X}\) be a solution of (2.1). Then \( \ell \) is nondecreasing.

**Proof.** Arguing by contradiction, assume there exists \( t > 0 \) such that \( \ell_t < L_t := \sup_{s \in [0, t]} \ell_s \). There are two possibilities.

1. There is \( s < t \) such that \( \ell_s = L_t \). In this case, \( \ell_\theta \leq \ell_s \) for all \( \theta \in [s, t] \). If for some \( x \) \( F(x) > \ell_s \) then \( F(x) > \ell_\theta \) for all \( \theta \in [s, t] \), and we can integrate (2.1)(i).

\[
u(x, t) = u(x, s) + \int_s^t \int_{\mathbb{R}^d} u(y, \theta)i(y, x)dyd\theta, \quad x \in F^{-1}(\ell_s, \infty).
\]

We obtain
\[
1 = \int_{\mathbb{R}^d} u(x, t)dx \geq \int_{F^{-1}(\ell_s, \infty)} \left[ u(x, s) + \int_s^t \int_{\mathbb{R}^d} u(y, \theta)i(y, x)dyd\theta \right]dx
\]
\[
\geq 1 + \int_s^t \int_{F^{-1}(\ell_s, \infty)} \int_{F^{-1}(\ell_s, \infty)} u(y, \theta)i(y, x)dydxd\theta.
\]
For every \( y \in F^{-1}(\ell_s, \infty) \), one has \( i(y) := \int_{F^{-1}(\ell_s, \infty)} i(y, x)dx > 0 \), by Assumption 2.1(ii).
Since \( \ell_s \geq \ell_\theta \) for all \( \theta \in [0, t] \), we have, similarly to (3.2), that
\[
u(y, \theta) = u_0(y) + \int_0^\theta \int_{\mathbb{R}^d} u(y', \theta')i(y', y)dy'd\theta', \quad y \in F^{-1}(\ell_s, \infty).
\]
Hence for such \( y \) and all \( \theta \in [s, t] \) one has \( u(y, \theta) \geq u_0(y) \). Hence the triple integral above is bounded below by

\[
(t - s) \int_{F^{-1}(\ell_s, \infty)} u_0(y) \psi(y) dy.
\]

But \( \int_{F^{-1}(\ell_s, \infty)} u_0(y) dy > 0 \) by assumption, hence the above integral is positive, a contradiction.

2. There is \( s \leq t \) such that \( \ell_{s-} = L_s \), and \( \ell_s < \ell_{s-} \). In this case there exists \( t' > s \) such that \( \ell_{\theta} \leq \ell_{s-} \) for all \( \theta \in [s, t'] \). We first show

\[
(3.3) \quad \int_{F^{-1}(\ell_s, \infty)} u(x, s) dx = 1.
\]

Let \( s_n \uparrow s \). Then \( \ell_{s_n} \to \ell_{s-} \) and by assumption, \( \ell_{s_n} \leq \ell_{s-} \). Now,

\[
(3.4) \quad \int_{F^{-1}(\ell_{s-}, \infty)} u(x, s) dx = \int_{F^{-1}(\ell_{s_n}, \infty)} u(x, s_n) dx + \int_{F^{-1}(\ell_{s_n}, \ell_{s-})} (u(x, s) - u(x, s_n)) dx
\]

\[-\int_{F^{-1}(\ell_{s_n}, \ell_{s-})} u(x, s) dx.
\]

The first term on the right is 1, and the last term converges to zero as \( n \to \infty \). As for the second term, since for all \( \theta \in [s_n, s] \) one has \( F^{-1}(\ell_{s-}, \infty) \subset F^{-1}(\ell_{\theta}, \infty) \), one can integrate in (2.1)(i) and get

\[
\int_{F^{-1}(\ell_{s-}, \infty)} |u(x, s) - u(x, s_n)| dx \leq \int_{s_n}^s \int_{\mathbb{R}^d} u(y, \theta) dy d\theta = s - s_n.
\]

The above expression and the second term in (3.4) have the same limit by the assumed boundedness of \( u \) on \( \mathbb{R}^d \times [0, s] \). This shows (3.3).

Using (3.3) and \( F^{-1}(\ell_{s-}, \infty) \subset F^{-1}(\ell_{t'}, \infty) \), we have \( \int_{F^{-1}(\ell_{t'}, \infty)} u(x, s) dx = 1 \). We can thus repeat the argument above in 1, with \((s, \ell_{s-}, t', \ell_{t'})\) in place of \((s, \ell_s, t, \ell_t)\). Namely,

\[
1 = \int_{F^{-1}(\ell_{t'}, \infty)} u(x, t') dx = \int_{F^{-1}(\ell_{t'}, \infty)} \left[ u(x, s) + \int_s^{t'} \int_{\mathbb{R}^d} u(y, \theta) \rho(y, x) dy d\theta \right] dx
\]

\[ \geq 1 + \int_s^{t'} \int_{\mathbb{R}^d} \int_{F^{-1}(\ell_{s-}, \infty)} u(y, \theta) \rho(y, x) dy dx d\theta. \]

The argument now completes exactly as in case 1. \( \square \)

**Lemma 3.2.** Let \((u, \ell)\) and \((v, m)\) be solutions of (2.1). Then \((u, \ell) = (v, m)\).

**Proof.** If \( w \in L^1(\mathbb{R}^d) \) and \( \int_{\mathbb{R}^d} w(x) dx = 0 \) then we have \( \| w \|_1 = 2 \| w^+ \|_1 = 2 \| w^- \|_1 \), where we denote \( \| \cdot \|_1 = \| \cdot \|_{L^1} \). If in addition \( w \geq 0 \) on some domain \( D \) then

\[
2 \| w^- \|_1 = 2 \| w^- 1_D \|_1 \leq 2 \| w 1_D \|_1.
\]

A similar statement holds with \( w \leq 0 \) and \( w^+ \). Hence if \( w \) is either nonnegative on \( D \) or nonpositive on \( D \),

\[
(3.5) \quad \| w \|_1 \leq 2 \| w 1_D \|_1.
\]
Consider \((x, t)\) such that \(F(x) > \ell_t\). Then \(F(x) > \ell_s\) for all \(s \leq t\). Therefore (2.1)(i) is valid with \((x, t)\) replaced by \((x, s)\) for all such \(s\), and

\[
u(x, t) = u_0(x) + \int_0^t \int_{\mathbb{R}^d} u(y, s) \rho(y, x) dy ds.
\]

Similarly, if \(F(x) > m_t\), the above is satisfied by \(v\). Denote \(\Delta = u - v\). Consider \((x, t)\) such that \(F(x) > \ell_t \lor m_t\). Then

\[
\Delta(x, t) = \int_0^t \int_{\mathbb{R}^d} \Delta(y, s) \rho(y, x) dy ds.
\]

Now, for each \(t\), \(\int_{\mathbb{R}^d} \Delta(x, t) dx = 1 - 1 = 0\). Moreover, in \(F^{-1}(-\infty, \ell_t \lor m_t)\), either \(u\) or \(v\) vanishes, therefore \(\Delta(\cdot, t)\) is either nonnegative or nonpositive. Hence we can apply (3.5) and then (3.6) to get

\[
\int_{\mathbb{R}^d} |\Delta(x, t)| dx \leq 2 \int_{F^{-1}(\ell_t \lor m_t, \infty)} |\Delta(x, t)| dx
\]
\[
\leq 2 \int_{F^{-1}(\ell_t \lor m_t, \infty)} \int_0^t \int_{\mathbb{R}^d} \Delta(y, s) |\rho(y, x)| dy ds dx
\]
\[
\leq 2 \int_0^t \int_{\mathbb{R}^d} |\Delta(y, s)| dy ds.
\]

The above is true for all \(t \geq 0\), hence by Gronwall’s lemma, the integral on the left vanishes for all \(t\). This shows that for every \(t\), \(u(x, t) = v(x, t)\) for a.e. \(x\). Next, for every \(t\), both \(u(\cdot, t)\) and \(v(\cdot, t)\) are continuous in each of the domains \(\{x \leq \ell_t \land m_t\}\), \(\{\ell_t \land m_t < x \leq \ell_t \lor m_t\}\) and \(\{x > \ell_t \lor m_t\}\), and therefore must be equal everywhere. This shows \(u = v\).

It remains to show that \(\ell = m\). Since \(u = v\), \((\ell, u)\) and \((m, u)\) are solutions. Arguing by contradiction, assume that, say, \(\ell_\theta < m_\theta\) for some \(\theta\). By right continuity, there exists an interval \([\theta, \theta_1]\) such that

\[
\sup_{t \in [\theta, \theta_1]} \ell_t < \inf_{t \in [\theta, \theta_1]} m_t.
\]

Let \(\hat{\ell}\) be defined by \(\hat{\ell}_t = m_t\) for \(t < \theta_1\) and \(\hat{\ell}_t = \ell_t\) for \(t \geq \theta_1\). Then \(\hat{\ell}\) is càdlàg. Moreover, the differential equation (2.1)(i) holds when \(F(x) > \hat{\ell}_t\) (because it holds in the larger domain \(F(x) > \ell_t\)), and the vanishing condition (2.1)(ii) holds when \(F(x) \leq \hat{\ell}_t\) (because it holds in the larger domain \(F(x) \leq m_t\)). Hence \((\hat{\ell}, u)\) is a solution. However, by construction, \(\hat{\ell}\) is not a nondecreasing trajectory, which contradicts the monotonicity property proved earlier. This shows that \(\ell = m\).

### 3.2. Proof of Theorem 2.2.

In this section it is proved first, in Lemma 3.3, that tightness holds, and then the main remaining task is to show that limits satisfy (2.1)(i). This is achieved by constructing upper and lower bounds on the density given in terms of limits of particle systems with piecewise constant boundary, for which convergence is a consequence of earlier work. These piecewise constant boundaries are constructed to form upper and lower envelopes of the prelimit free boundary \(\ell^N\). (As already mentioned in the introduction, these discrete approximations do not form barriers in the sense of [8] and other literature on the subject, in that they are constructed from a given solution, and therefore do not suffice for a uniqueness argument).
Lemma 3.3. The sequence of laws of $(\xi^N, \ell^N)$ is $C$-tight. Moreover, every subsequential limit $(\xi, \ell)$ satisfies a.s.,

$$\xi_t(F^{-1}(-\infty, \ell_t)) = 0, \text{ for all } t \in (0, \infty).$$

Proof. We first argue $C$-tightness of $\ell^N$. Since $J^N$ is a rate-$N$ Poisson process, $\tilde{J}^N$ converges in probability to the identity map from $\mathbb{R}^d$ to itself. Next, by construction, the path $t \mapsto \ell^N_t$ is nondecreasing, because when a new particle is born at $t$ in the domain $F^{-1}[\ell^N_{t-}, \infty)$ one has $\ell^N_t \geq \ell^N_{t-}$, and if it is born outside this domain, it is removed immediately and $\ell^N_t = \ell^N_{t-}$. In particular, $\ell^N_t \geq \ell^N_0 \to \lambda_0$ in probability. Fix $T > 0$. We show that the random variables $\ell^N_T$ are tight. To this end, denote by $\tilde{\xi}^N$ the configuration measure associated with non-local branching without removals. That is, they are constructed as our original system, but without removing any particles, and the systems are coupled so that at all times, the configuration of the original system is a subset of that of the enlarged one. Then there exists a (deterministic) finite measure on $\mathbb{R}^d$, $\zeta_T$, such that $\tilde{\xi}^N_T \to \zeta_T$ in probability by [16, Theorem 5.3]. As a result, there exists a compact $K \subset \mathbb{R}^d$ such that $\limsup_N \mathbb{P}(\tilde{\xi}^N_T(K^c)) > \frac{1}{2} = 0$. Let $k = \max\{F(x) : x \in K\}$. Then $\limsup_N \mathbb{P}(\tilde{\xi}^N_T(F^{-1}(k, \infty)) > \frac{1}{2}) = 0$. Since $\tilde{\xi}^N_T$ is dominated by $\zeta^N_T$ a.s.,

$$\mathbb{P}(\tilde{\xi}^N_T > k) = \mathbb{P}(\tilde{\xi}^N_T(F^{-1}(k, \infty)) \leq N) \leq \mathbb{P}(\tilde{\xi}^N_T(F^{-1}(k, \infty)) \geq 1),$$

showing $\limsup_N \mathbb{P}(\tilde{\xi}^N_T > k) = 0$. Next, for $\ell^N_T$ to increase over a time interval $[t, t + h]$ by more than $\delta$, the particles located at time $t$ in $D^N(t, \delta) := F^{-1}[\ell^N_t, \ell^N_t + \delta]$ must be removed by time $t + h$. Because $\ell^N_t$ is monotone, all particles in the initial configuration within the domain $D^N(t, \delta)$ are still present at time $t$. Hence the event $\ell^N_{t+h} > \ell^N_t + \delta$ is contained in

$$\tilde{\xi}^N_0(D^N(t, \delta)) \leq \tilde{J}^N_{t+h} - \tilde{J}^N_t.$$

As a result, the event $w_T(\ell^N, h) > \delta$ is contained in

$$\inf_{t \in [0, T-h]} \tilde{\xi}^N_0(D^N(t, \delta)) \leq w_T(\tilde{J}^N_t, h).$$

Hence for any $\delta, h, \varepsilon > 0$,

$$\mathbb{P}(w_T(\ell^N, h) > \delta) \leq \mathbb{P}(\ell^N_T > k) + \mathbb{P}(w_T(J^N, h) > \varepsilon))$$

$$+ \mathbb{P}\left( \inf_{t \in [0, T-h]} \tilde{\xi}^N_0(D^N(t, \delta)) \leq \varepsilon, \ell^N_T \leq k \right).$$

Thus

$$\limsup_N \mathbb{P}(w_T(\ell^N, h) > \delta) \leq \limsup_N \mathbb{P}(w_T(J^N, h) > \varepsilon))$$

(3.7)

$$+ \limsup_N \mathbb{P}\left( \inf_{a \in [\lambda_0, k]} \tilde{\xi}^N_0(F^{-1}[a, a + \delta]) < \varepsilon \right).$$

Let $\delta > 0$ be given. By assumption, $\eta(a, \delta) := \int_{\mathbb{R}^d \setminus (a, a + \delta)} u_0(x) dx > 0$, $a \geq \lambda_0$. Moreover, by the boundedness of $u_0$, $a \mapsto \eta(a, \delta)$ is continuous. Therefore $\inf_{a \in [\lambda_0, k]} \eta(a, \delta) > 0$. This and the convergence $\tilde{\xi}^N_0 \to \xi_0 = u_0(x) dx$ in probability show that one can find $\varepsilon > 0$ such that the second term on the r.h.s. of (3.7) vanishes. Given such $\varepsilon$, using the fact that limits of $J^N$ are 1-Lipschitz, the first on the r.h.s. of (3.7) also vanishes provided $h$ is sufficiently small. This completes the proof of $C$-tightness of $\ell^N$.

Next, $C$-tightness of $\tilde{\xi}^N$ is shown. Denote by $d_L$ the Levy-Prohorov metric on $\mathcal{P}(\mathbb{R}^d)$, which is compatible with weak convergence on this space. We will show that (i) for every
\( \varepsilon > 0 \) and \( t \) there exists a compact set \( K_{\varepsilon, t} \subset \mathcal{P}(\mathbb{R}^d) \) such that \( \liminf_N \mathbb{P}(\xi^N_t \in K_{\varepsilon, t}) > 1 - \varepsilon \); and (ii) for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[
\limsup_N \mathbb{P}(w_T(\xi^N, \delta) > \varepsilon) < \varepsilon.
\]
This will establish \( C \)-tightness of \( \xi^N_t \) in view of [15, Corollary 3.7.4 (p. 129)] and because we use \( w \) rather than \( w' \) [15, (3.6.2) (p. 122)].

To show (i), let
\[
K_n(r) = \{ \gamma \in \mathcal{P}(\mathbb{R}^d) : \gamma(\mathbb{B}_r^c) < n^{-1} \}, \quad n \in \mathbb{N}, r \in (0, \infty).
\]
By Prohorov's theorem, for any \( n_0 \) and sequence \( \{ r_n \} \), the closure of \( K_{\geq n_0}(\{ r_n \}) := \bigcap_{n \geq n_0} K_n(r_n) \), as a subset of \( \mathcal{P}(\mathbb{R}^d) \), is compact. Suppose we show that there exists a sequence \( r_n \) such that, for every \( t \in [0, T] \),
\[
(3.8) \quad \liminf_N \mathbb{P}(\xi^N_t \in K_n(r_n)) \geq 1 - 2^{-n}.
\]
Then, given \( \varepsilon > 0 \), taking \( n_0 \) such that \( \sum_{n \geq n_0} 2^{-n} < \varepsilon \), it would follow that
\[
\liminf_N \mathbb{P}(\xi^N_t \in K_{\geq n_0}(\{ r_n \})) > 1 - \varepsilon,
\]
showing that (i) holds. To this end, note that the convergence of \( \xi^N_t \) implies that there exists a sequence \( \{ r_n \} \) such that
\[
\liminf_N \mathbb{P}(\xi^N_{T_n}(\mathbb{B}^c) < n^{-1}) > 1 - 2^{-n}.
\]
But
\[
\sup_{t \in [0, T]} \xi^N_t(\mathbb{B}^c) \leq \xi^N_{T_n}(\mathbb{B}^c),
\]
hence (3.8) follows, and (i) is proved.

Next we show (ii). Given a set \( C \subset \mathbb{R} \) let \( C^\varepsilon \) denote its \( \varepsilon \)-neighborhood. Let \( 0 \leq s < t \leq T \) be such that \( t - s \leq \delta \). Then for any Borel set \( C \), one has
\[
|\xi^N_t(C) - \xi^N_s(C)| \leq J^N_t - J^N_s.
\]
Hence, on the event \( w_T(\bar{J}^N, \delta) \leq \varepsilon \),
\[
\bar{\xi}^N_s(C) \leq \bar{\xi}^N_t(C^\varepsilon) + \varepsilon, \quad \text{and} \quad \bar{\xi}^N_t(C) \leq \bar{\xi}^N_s(C^\varepsilon) + \varepsilon,
\]
and thus \( d_L(\bar{\xi}^N_s, \bar{\xi}^N_t) \leq \varepsilon \). This shows that for sufficiently small \( \delta \),
\[
\limsup_N \mathbb{P}(w_T(\bar{\xi}^N, \delta) > \varepsilon) \leq \varepsilon,
\]
and the proof of (ii) is complete.

For the second assertion of the lemma, note that by the definition of \( \ell^N \), one has for all \( N \),
\[
\xi^N_t(F^{-1}(-\infty, \ell^N)) = 0.
\]
Invoking Skorohod's representation, one has \( (\bar{\xi}^N_t, \ell^N) \to (\xi, \ell) \) a.s. along the convergent subsequence. Hence for \( t > 0 \) and \( \varepsilon > 0 \), because \( F^{-1}(-\infty, \ell - \varepsilon) \) is open,
\[
\xi_t(F^{-1}(-\infty, \ell_t - \varepsilon)) \leq \liminf_N \bar{\xi}^N_t(F^{-1}(-\infty, \ell_t - \varepsilon)) \leq \liminf_N \bar{\xi}^N_t(F^{-1}(-\infty, \ell^N)) = 0,
\]
a.s. Taking \( \varepsilon \downarrow 0 \), \( \xi_t(F^{-1}(-\infty, \ell_t)) = 0 \), a.s. To deduce the a.s. statement simultaneously for all \( t \), it suffices to note that for \( t_n \downarrow t \), by monotonicity of \( \ell \), one has
\[
\xi_t(F^{-1}(-\infty, \ell_t)) \leq \liminf_n \xi_{t_n}(F^{-1}(-\infty, \ell_t)) \leq \liminf_n \xi_{t_n}(F^{-1}(-\infty, \ell_{t_n})) = 0.
\]
PROOF OF THEOREM 2.2. In view of Lemmas 3.2 and 3.3, the proof will be complete once it is shown that for every limit \((\xi, \ell)\) there exists a measurable density \(u\) such \((u, \ell, t) \in X\) and \((u, \ell)\) satisfies (2.1). Fix a convergent subsequence and denote its limit by \((\xi, \ell)\).

To argue the existence of a density let us go back to the particle system with no removals, mentioned in the proof of Lemma 3.3. The normalized process \(\bar{\zeta} \rightarrow \zeta\), in probability, where \(\zeta\) is deterministic and for every \(t\) \((\zeta_t)\) once it is shown that for every limit \(\zeta_t \rightarrow \zeta\) \((\xi_t)\) has a density, as follows from [16, Theorem 5.3 and Proposition 5.4]. Throughout what follows, denote this density by \(x(\cdot, t)\).

For any bounded continuous \(g : \mathbb{R}^d \rightarrow [0, \infty)\), one has \(E \int g(x)\zeta_t^N(dx) \leq E \int g(x)\bar{\zeta}_t^N(dx), \) \(t \in [0, T]\). Hence \(E \int g(x)\xi_t(dx) \leq E \int g(x)\zeta_t(dx), \) \(t \in [0, T]\). In particular, \(\xi_t(dx) \leq \xi_t, t \leq T, \) and since \(T\) is arbitrary, this statement holds for all \(t\). Assuming \(\xi = 0\) outside the full measure event, we finally obtain that for every \((t, \omega) \in (0, \infty) \times \Omega, \) \(\xi_t(dx, \omega) \leq dx\). We can now appeal to [12, Theorem 58 in Chapter V (p. 52)] and the remark that follows. The measurable spaces denoted in [12] by \((\Omega, \mathcal{F})\) and \((T, \mathcal{T})\) are taken to be \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\) and \(((0, \infty) \times \Omega, \mathcal{B}((0, \infty)) \otimes \mathcal{F})\), respectively. According to this result there exists a \(\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}((0, \infty)) \otimes \mathcal{F}\)-measurable function \(u(x, t, \omega)\), such that for every \((t, \omega) \in (0, \infty) \times \Omega, u(\cdot, t, \omega)\) is a density of \(\xi_t(dx, \omega)\) with respect to \(dx\). We also have \(u(x, t, \omega) \leq z(x, t)\).

Items (ii), (iii) and (iv) of (2.1) can be verified plainly: In view of Lemma 3.3, \(u\) has a version satisfying \(u(x, t) = 0\) for all \((x, t)\) such that \(F(x) < \ell_t\), and this extends to \(F(x) \leq \ell_t\) using Assumption 2.1(i) by which \(\text{Leb}F^{-1}\{\ell_t\} = 0\) for all \(t\). This verifies (2.1)(ii). Items (iii) and (iv) are obvious from our assumptions. It remains to show (2.1)(i), which by the nondecreasing property of \(\ell\) can be expressed as

\[
(3.9) \quad u(x, t) = u_0(x) + \int_0^t \int_{\mathbb{R}^d} u(y, s)\rho(y, x)dyds, \quad (x, t) : F(x) > \ell_t, t > 0.
\]

Note that as soon as the above is established, the uniform continuity of \(\rho\) and the local integrability of \(u\) imply the existence of a version of \(u\) for which \(x \mapsto u(x, t)\) and \(t \mapsto \partial_t u(x, t)\) are continuous, hence \((u, \ell) \in X\). We will achieve (3.9) in several steps.

Step 1. Integro-differential systems with piecewise constant boundary. Fix \(T > 0\) throughout. For \(\delta, \varepsilon \in (0, 1)\) such that \(J := \delta^{-1}T \in \mathbb{N}\), let \(M = M(\delta, \varepsilon)\) denote the collection of right-continuous piecewise constant nondecreasing trajectories \(m : [0, T) \rightarrow \mathbb{R}\) such that, with \(t_j = j\delta\), for each \(j = 0, 1, \ldots, J - 1\), on \([t_j, t_{j+1})\), \(m\) takes a constant value in \(\varepsilon\mathbb{Z}\). Denote \(a_j(m) = m(t_j)\). For \(m \in M\) and \(a_j = a_j(m)\), consider the set of equations

\[
(3.10) \quad \begin{cases}
\partial_t u(x, t) = \int_{\mathbb{R}^d} u(y, t)\rho(y, x)dy & (x, t) : F(x) > a_j, t \in (t_j, t_{j+1}), \\
u(x, t) = 0 & (x, t) : F(x) \leq a_j, t \in (t_j, t_{j+1}), \\
u(x, t_j) = u(x, t_j - 1_{\{F^{-1}[a_j, \infty)\}}(x) & 1 \leq j \leq J - 1, \\
u(x, 0) = u_0(x)1_{\{F^{-1}[a_0, \infty)\}}(x).
\end{cases}
\]

It is clear, by induction, that this set uniquely determines a solution, denoted \(u^{(m)}\). If \(\zeta_t^{(m)}(dx) = u^{(m)}(x, t)dx, t \in [0, T]\), then the map \(M \ni m \mapsto \zeta^{(m)} \in D([0, T), \mathcal{M}_+(\mathbb{R}^d))\) is denoted by \(\mathcal{Z}\).

Step 2. A continuity property of \(\mathcal{Z}\). We prove the following claim. There exists a function \(\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+\) such that \(\gamma(0+) = 0\) and the following holds. Let \(k, m \in M\) and \(w = u^{(m)}, \) \(v = u^{(m)}\). Denote \(\Delta = w - v\). Then

\[
(3.11) \quad \sup_{t < T} \|\Delta(\cdot, t)\|_1 \leq \gamma(\|m - k\|_T), \quad k, m \in M.
\]
To prove the claim, denote
\[ \eta(k, m) = \max_{j \leq J-1} \zeta_{\Delta}(F^{-1}(a_j(k) \land a_j(m)), a_j(k) \lor a_j(m)). \]

Note that
\begin{align*}
(3.12) & \quad w(x, t) = w_0(x) + \int_0^t \int_{\mathbb{R}^d} w(y, s) \rho(y, x) dy ds, \quad (x, t) : F(x) > k_t, t > 0, \\
(3.13) & \quad v(x, t) = v_0(x) + \int_0^t \int_{\mathbb{R}^d} v(y, s) \rho(y, x) dy ds, \quad (x, t) : F(x) > m_t, t > 0,
\end{align*}

and \( w \) (resp., \( v \)) vanishes to the left of \( k \) (resp., \( m \)). Also note that \( w(x, t) \lor v(x, t) \leq z(x, T) \).

For \( t \in [0, T) \),
\[ \|\Delta(\cdot, t)\|_1 \leq \|\Delta(\cdot, 0)\|_1 + \int_{F^{-1}(k_t, \infty) \Delta F^{-1}(m_t, \infty)} (w \lor v)(x, t) dx \]
\[ + \int_{F^{-1}(k_t, \infty) \Delta F^{-1}(m_t, \infty)} \int_0^t |\Delta(y, s)| \rho(y, x) dy ds dx \]
\[ \leq \|\Delta(\cdot, 0)\|_1 + \eta(k, m) + \tilde{\epsilon} \int_0^t \|\Delta(\cdot, s)\|_1 ds. \]

Since \( \|\Delta(\cdot, 0)\|_1 \leq \eta(k, m) \), Gronwall’s lemma gives
\[ \|\Delta(\cdot, t)\|_1 \leq 2\eta(k, m)e^{\tilde{\epsilon}T}, \quad t < T. \]

Hence (3.11) will be proved once it is shown that \( U(0+) = 0, \) where
\[ U(\kappa) = \sup \{ \zeta_{\Delta}(F^{-1}(b, b + \kappa)) : b \in \mathbb{R} \}, \quad \kappa > 0. \]

Assuming the contrary, there exists \( \kappa' > 0 \) and \( \{b_n\} \) such that, along a subsequence,
\[ (3.14) \quad \zeta_{\Delta}(F^{-1}(b_n, b_n + n^{-1})) > \kappa'. \]

Now, \( \{b_n\} \) must be bounded. For if there is a subsequence with \( b_n \to \infty \), by the continuity of \( F \), for every compact \( K \) the set \( F^{-1}(b_n, \infty) \cap K \) must be empty for large \( n \), which shows that (3.14) cannot hold. If there is a subsequence \( b_n \to -\infty \) then \( F^{-1}(\infty, b_n + n^{-1}) \cap K \) must be empty for all large \( n \) and again (3.14) cannot hold. Hence \( b_n \) is bounded. Let \( b \) be a limit. Then, given \( \kappa_1 > 0, (b_n, b_n + n^{-1}) \subset (b - \kappa_1, b + \kappa_1) \) for large \( n \) along a subsequence, showing
\[ \kappa' \leq \zeta_{\Delta}(F^{-1}(b - \kappa_1, b + \kappa_1)). \]

Hence \( \zeta_{\Delta}(F^{-1}\{b\}) > 0 \), contradicting our assumption \( \text{Leb}(F^{-1}\{a\}) = 0 \) for every \( a \). This proves (3.11).

Step 3. Particle systems with piecewise constant boundary. Given \( m \in M \) we construct a particle system in which, for every \( t \in [0, T] \), all particles lie within \( F^{-1}[m_t, \infty) \). Its initial configuration is given by the restriction of \( \{x_i\} \) to \( F^{-1}[m_0, \infty) \). During any interval \( (t_j, t_{j+1}) \), the reproduction is according to \( \rho \), but every newborn outside of \( F^{-1}[a_j(m), \infty) \) is immediately removed. If for \( j \in \{1, \ldots, J - 1\}, t_j \) is a continuity point of \( m \) nothing happens at this time. Otherwise, \( m \) necessarily performs a positive jump, hence the domain \( F^{-1}[m, \infty) \) decreases. At this time, all particles outside \( F^{-1}[a_j(m), \infty) \) are removed. Denote the configuration process by \( \zeta^{(m), N} \).

We show that for every \( m \in M \), \( \zeta^{(m), N} \to \zeta^{(m)} := \mathcal{Z}(m) \) in probability, uniformly on \([0, T)\). For the initial condition, we have \( \zeta_0^{(m), N} = \zeta_0^{N 1_{F^{-1}[a_0, \infty)}} \). The boundary of the
domain $F^{-1}(a_0, \infty)$ has Lebesgue measure zero, as follows from Assumption 2.1(i) upon noting that $\partial F^{-1}(a, \infty) \subset F^{-1}(a)$ for every $a \in \mathbb{R}$. Because the limit $\xi_0$ has a density, this implies $\zeta_0^{(m), N} \rightarrow \xi_0 1_{\{F^{-1}(a_0, \infty)\}}$ in probability.

Assume that for $j \geq 0$, $\zeta_j^{(m), N} \rightarrow \zeta_j^{(m)}$. Then the convergence on the interval $(t_j, t_{j+1})$ to a measure-valued trajectory which has a density satisfying the integro-differential equation in (3.10) is as in the proof of [13, Proposition 2.1]; the proof there, based on [16], is for $d = 1$ and the domain $\mathbb{R}$, but the same proof is valid for $d \geq 1$ an arbitrary domain of $\mathbb{R}^d$, because that is the generality of the results of [16]. Because the trajectory is continuous on $(t_j, t_{j+1})$, the convergence is uniform there. Next, at the trimming time $t_{j+1}$, the convergence follows by the same argument as for $t = 0$.

Step 4. Relation to the original particle system. Consider now, for each $N$, two stochastic processes, $k^N$ and $m^N$, with sample paths in $M$, defined as follows. For $x \in \mathbb{R}$, let $P^-_\varepsilon(x) = \max\{y \in \varepsilon Z : y \leq x\}$, $P^+_\varepsilon(x) = \min\{y \in \varepsilon Z : y \geq x\}$, and let

$$k^N_t = P^-_\varepsilon \left( \inf_{s \in [t_j, t_{j+1}]} ^N \right) - \varepsilon, \quad m^N_t = P^+_\varepsilon \left( \sup_{s \in [t_j, t_{j+1}]} ^N \right) + \varepsilon, \quad t \in [t_j, t_{j+1}).$$

By construction,

$$k^N_t < \ell^N_t < m^N_t, \quad t \in [0, T).$$

Constructing a particle system with piecewise constant boundary, as in Step 2, makes perfect sense even when the trajectories are random members of $M$, specifically, $k^N$ and $m^N$. To construct these systems, one first evaluates $\ell^N_t$, and based on it, $k^N$ and $m^N$, and then uses these random trajectories to construct the particle systems as in Step 2. A key point is that, thanks to (3.15), one can couple the three systems in such a way that the ordering

$$\zeta_t^{(m), N} \subset \zeta_t^{N} \subset \zeta_t^{(k), N}, \quad t \in [0, T),$$

is maintained a.s. Above, we have denoted the two particle configurations corresponding to the piecewise constant boundaries by $\zeta^{(k), N}$ and $\zeta^{(m), N}$. In this coupling, one must be careful to use, in the construction of $\zeta^{(m), N}$ and $\zeta^{(k), N}$, the same stochastic primitives (namely, exponential clocks and birth locations) that were used in the construction of $(\ell^N, \ell^N)$. Relation (3.16) can then be shown by arguing that every particle existing in the $\zeta^{(m), N}$ (resp., $\zeta^{N}$) configuration at any given time also exists at the $\zeta^{N}$ (resp., $\zeta^{(k), N}$) configuration at that time. Showing that this is true during the intervals $(t_j, t_{j+1})$ proceeds by induction on the times when particles are born, whereas at the times $t_j$, one notices that the trimming operation preserves the order as long as (3.15) holds. The construction is straightforward and we therefore skip its precise details.

Since the maps $P^\pm_\varepsilon$ are not continuous, $k^N$ and $m^N$ may not converge in law. However, the tightness of the laws of $\ell^N_t$ and the specific structure of $M$ clearly give tightness of $(k^N, m^N)$. Consider then any convergent subsequence $(\xi^N_j, \ell^N_j, k^N_j, m^N_j) \Rightarrow (\xi, \ell, \hat{k}, \hat{m})$. We claim that the limit in law of $(\xi^N_j, \ell^N, k^N_j, m^N_j, \zeta^{(k), N}_j, \zeta^{(m), N}_j)$ exists and is given by

$$(\xi, \ell, \hat{k}, \hat{m}, Z(\hat{k}), Z(\hat{m})).$$

To simplify the notation, we prove the claim only for the 3-tuple $(\xi^N_j, m^N_j, \zeta^{(m), N}_j)$; the proof for the 6-tuple is very similar. Because the claim is concerned with convergence of probability measures, it suffices to prove vague convergence. Let $f : D_T \rightarrow \mathbb{R}$ be compactly supported, where $D_T := D([0, T], M(\mathbb{R}^d)) \times D([0, T], \mathbb{R}) \times D([0, T], M(\mathbb{R}^d))$. Let $M_f$ be the set of $m \in M$ for which $f(\cdot, m, \cdot) \neq 0$ and note that $\#M_f < \infty$ because for $c > 0$ there
are only finitely many \( m \in M \) with \( \|m\|_T < c \). By the uniform continuity of \( f \), one has 
\[ \gamma_f(0+) = 0, \]
where
\[ \gamma_f(\kappa) = \sup\{|f(\alpha, m, \beta) - f(\alpha, m, \beta')| : (\alpha, m, \beta) \in D_T, d_L(\beta, \beta') < \kappa\}. \]

Now,
\[
\mathbb{E} f(\xi^N, m^N, \zeta^{(m)N}) = \sum_{m \in M_T} \mathbb{E}[1(m)\{m^N\}] f(\xi^N, m, \zeta^{(m)N})
\]
\[
= \sum_{m \in M_T} \mathbb{E}[1(m)\{m^N\}] f(\xi^N, m, \mathcal{Z}(m)) + W^N,
\]
where
\[
|W^N| \leq \sum_{m \in M_T} \mathbb{E}[\gamma_f(d_L(\zeta^{(m)N}, \mathcal{Z}(m)))] \to 0,
\]
by the convergence in probability proved in Step 3. Because each \( m \in M \) is isolated from all other members of \( M \), the convergence \( (\xi^N, m^N) \Rightarrow (\xi, \hat{m}) \) implies \( \mathbb{E}[1(m)\{m^N\}] \to \mathbb{E}[1(\hat{m})\{\hat{m}\}] \) whenever \( g \) is bounded and continuous. This gives
\[
\mathbb{E} f(\xi^N, m^N, \zeta^{(m)N}) \to \sum_{m \in M_T} \mathbb{E}[1(\hat{m})\{\hat{m}\}] f(\xi, m, \mathcal{Z}(\hat{m})) = \mathbb{E}[f(\xi, \hat{m}, \mathcal{Z}(\hat{m}))],
\]
proving the claim.

As a consequence of the above convergence and (3.16), we obtain a bound on the density \( u \), namely, for every \( t \) and a.e. \( x \),
\[ (3.17) \quad \hat{u}(x, t) \leq u(x, t) \leq \check{u}(x, t), \]
where \( \hat{u} = u^{(\hat{m})} \) and \( \check{u} = u^{(\hat{k})} \) are the random densities corresponding to \( \mathcal{Z}(\hat{m}) \) and \( \mathcal{Z}(\hat{k}) \).

Step 5. Completing the proof. Denoting \( \hat{\Delta} = u - \hat{u}, \check{\Delta} = \check{u} - \hat{u} \), and \( \check{q} = \|\hat{m} - \hat{k}\|_T^\gamma \), we have by (3.17), \( \|\hat{\Delta}(\cdot, t)\|_1 \leq \|\check{\Delta}(\cdot, t)\|_1 \leq \gamma(\check{q}) \), where (3.11) is used.

Note that \( \check{u} \) and \( \hat{u} \) satisfy (3.12)–(3.13), with \( k \) and \( m \) replaced by the random \( \hat{k} \) and \( \hat{m} \). Let \( (x, t) \) be such that \( F(x) > \hat{m}_t \), by which \( F(x) > \ell_t \geq \check{k}_t \). Then
\[
\hat{u}(x, t) = \hat{u}(x, t) + \Delta(x, t)
\]
\[
= u_0(x) - \Delta_0(x) + \int_0^t \int_{\mathbb{R}^d} u(y, s)\rho(y, x)dyds - \int_0^t \int_{\mathbb{R}^d} \Delta(y, s)\rho(y, x)dyds + \Delta(x, t).
\]

Denoting, for \( (x, t) \in \mathbb{R}^d \times [0, T) \),
\[
\Psi(x, t) = \left| u(x, t) - u_0(x) - \int_0^t \int_{\mathbb{R}^d} u(y, s)\rho(y, x)dyds \right|,
\]
we therefore have
\[
\int_{\mathbb{R}^d} \Psi(x, t)1_{\{F(x) > \hat{m}_t\}}dx \leq (2 + \check{c}T)\gamma(\check{q}).
\]
Moreover,
\[
\Psi(x, t) \leq 2z(x, T) + T \int_{\mathbb{R}^d} z(y, T)\check{\rho}(x - y)dyds.
\]
Analogously to $U$, define $\tilde{U}(\kappa) = \sup \{ \int_{\mathbb{R}} \Psi(x,t) 1_{\{F(x) \leq \hat{m}_t\}} \ dx : b \in \mathbb{R} \}$, and note, by similar reasoning, that $\tilde{U}(0+) = 0$. Then

$$\int_{\mathbb{R}^d} \Psi(x,t) 1_{\{\ell, \delta < F(x) \leq \hat{m}_t\}} \ dx \leq 2U(\hat{q}) + T\tilde{c}|\zeta_T|\tilde{U}(\hat{q}).$$

Hence there exists a constant $c$ such that

$$\psi := \sup_{t < T} \int_{\mathbb{R}^d} \Psi(x,t) 1_{\{F(x) > \ell, \delta \}} \ dx \leq c(\gamma(\hat{q}) + U(\hat{q}) + \tilde{U}(\hat{q})).$$

Given $\epsilon, \delta' > 0$ choose $\delta > 0$ so small that $\mathbb{P}(\Omega_{\delta, \epsilon}) > 1 - \delta'$, where $\Omega_{\delta, \epsilon} = \{ w_T(\ell, \delta) < \epsilon \}$. By the definition of $m^N$, one has for every $j$, $m^N_{t_j} \leq \sup_{[t_j, t_{j+1}]} \ell^N + 2\epsilon$, hence $\hat{m}_t \leq \sup_{[t_j, t_{j+1}]} \ell + 2\epsilon$. An analogous statement holds for $\hat{k}$. This gives, on $\Omega_{\delta, \epsilon}$, $\hat{m}_t - \hat{k}_t \leq 5\epsilon$ for all $t < T$, hence $\hat{q} \leq 5\epsilon$. As a result,

$$\mathbb{P}(\psi > c(\gamma(5\epsilon) + U(5\epsilon) + \tilde{U}(5\epsilon))) < \delta'.$$

Finally, $\delta' \to 0$ followed by $\epsilon \to 0$ shows $\psi = 0$ a.s. This shows (3.9) and completes the proof.

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