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A WEAK FORMULATION OF FREE BOUNDARY PROBLEMS AND ITS APPLICATION TO HYDRODYNAMIC LIMITS OF PARTICLE SYSTEMS WITH SELECTION

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A weak formulation for a class of parabolic free boundary problems (FBP) is proposed that does not involve the notion of a free boundary but reduces to a FBP when classical solutions exist. It is aimed at hydrodynamic limits (HDL) of particle systems with selection in circumstances where the macroscopic model does not possess (or is hard to prove to possess) a regular free boundary in the classical sense. The formulation involves the macroscopic density of particles and a measure that accounts for selection. It consists of a second- order parabolic equation satisfied by the density and driven by the measure, coupled with a complementarity condition satisfied by the density-measure pair. The approach is applied to an injection-branching-selection particle system of diffusion on $\mathbb{R}$ under arbitrarily varying injection and removal rates, for which the corresponding FBP is not in general known to be classically solvable. The HDL is characterized as the unique solution to the weak formulation. The proof of convergence is based on PDE uniqueness, which in turn relies on the barrier method.

1. Introduction.

1.1. *Background and motivation.* In particle systems with spatial selection, particles living in $\mathbb{R}$ undergo motion, branching or injection and selection. The last term refers to keeping the population size constant by removing, upon appearance of a new particle, the particle whose position is smallest among all particles (henceforth, the leftmost particle). The first such systems were proposed in [6, 7] as models for natural selection in the evolution of a population, where the position of a particle represents the degree of fitness of an individual to its environment. A series of papers culminating in the monograph [10] studied a variety of related models motivated by particles interacting topologically (since, macroscopically, removals occur at the boundary of the configuration) and by particle systems in contact with current reservoirs. At the hydrodynamic limit (HDL), these models give rise to free boundary problems (FBP). Rigorously establishing the HDL–FBP relation requires control over regularity of the free boundary (such as C^1 or sometimes C). We are motivated by questions of characterizing HDL by PDE in cases where existing techniques might fall short of yielding free boundary regularity. This may happen, in particular, when the constant population size assumption is dropped and injection and removal rates vary at the macroscopic scale. The first goal of this paper is to introduce a weak formulation of FBP that does not involve the notion of a free boundary, and at the same time reduces to a classical FBP when classical solutions exist.

To put these questions in context, consider the *N-particle branching Brownian motion* (N-BBM) in dimension 1, first studied in [23], which consists of N particles performing Brownian motion (BM) independently, each branching into two at rate 1. When branching

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occurs, the leftmost particle in the configuration is removed. The initial particle positions are drawn independently according to a probability measure ξ_0. Let $\xi^N_t = \sum_i \delta_{X_i(t)}$ denote the configuration measure at time t, where $X_i(t)$ are the locations of the N particles alive at time t. Here and throughout, δ_x denotes the Dirac measure at x. Throughout, the bar notation stands for normalization by N; in particular, $\bar{\xi}^N_t = N^{-1}\xi^N_t$. The corresponding FBP is to find a pair (u, ℓ), $\ell \in C((0,\infty) : \mathbb{R})$, $u \in C(\mathbb{R} \times (0,\infty) : [0,\infty)) \cap C^{2,1}(\{(x,t) : t > 0, x > \ell_t\})$ such that

$$
(1.1)\qquad \begin{cases} \partial_t u - \frac{1}{2}\partial_x^2 u = u, & x > \ell_t, \\ u = 0, & x \le \ell_t, \\ u(x,t)\,dx \to \xi_0(dx), & \text{weakly as } t \downarrow 0, \\ \int_{\mathbb{R}} u(x,t)\,dx = 1. \end{cases}
$$

It was shown in [12] (for ξ_0 possessing a density) that the process $\bar{\xi}^N_t$ has a deterministic limit, characterized in terms of barriers (see Section 3). Under the assumption that (1.1) has a classical solution and that the free boundary ℓ is C^1, it was further shown that the limit process has a density given by the unique solution to (1.1). In [4], it was then shown, for general ξ_0, that (1.1) has a unique classical solution and that the limit of $\bar{\xi}^N$ has a density given by u (with ℓ only in C). In [10], a model we will refer to as the *injection-selection* model was studied, in which a collection of N Brownian particles living in $\mathbb{R}_+$ and reflecting at the origin, is subject to injection of new particles at the origin, at times determined by a rate-c_0N exponential clock, $c_0 > 0$ a constant. Upon each injection, the rightmost particle is removed (i.e., the one whose location on $\mathbb{R}$ is greatest). The corresponding FBP is to find (u, ℓ) such that

$$
(1.2)\qquad \begin{cases} \partial_t u - \frac{1}{2}\partial_x^2 u = 0, & 0 < x < \ell_t, \\ u = 0, & x \ge \ell_t, \\ -\frac{1}{2}\partial_x u(0,t) = -\frac{1}{2}\partial_x u(\ell_t,t) = c_0, & \\ u(\cdot,0) = u_0, & \end{cases}
$$

with u_0 an initial density. It is shown there that the HDL exists and possesses a density. Moreover, to overcome questions of regularity of the free boundary, a weak formulation of solutions to (1.2) is proposed there, defined via approximations by local classical solutions, and it is proved that such a solution uniquely exists and is equal to the aforementioned HDL density (see Section 1.3 for more details).

The context in which the weak formulation is presented, in Section 1.2 below, is an *injection-branching-selection* system of diffusion processes on $\mathbb{R}$, which extends both the aforementioned ones (except the minor detail that, in [10], particles live in $\mathbb{R}_+$). In this model, the mass conservation condition is abandoned, and the rates of injection and removal of mass may vary. Such a perturbation has dramatic consequences on the macroscopic model to the extent that they may lead to high degree of free boundary irregularity (see Remark 2.6). It is not clear whether the current toolboxes of either the classical solution approach of [4] or the weak solution approach of [10] can potentially cover such scenarios. We will show that the weak formulation introduced here does.

Our second goal has to do with the applicability of the PDE uniqueness approach to studying HDL, which consists of showing that all limit laws are supported on solutions to a PDE that possesses a unique solution. The use of this approach has been missing from the literature on the subject, precisely due to difficulties regarding free boundary regularity; we refer to [11] for a discussion of this point. We will show that the weak formulation fills this gap at least insofar as the injection-branching-selection model is concerned.

1.2. *Injection-branching-selection and weak formulation.* A brief description of the model is as follows. Brownian particles on the line, whose initial number is N, branch at rate $\kappa \geq 0$. In addition, injections occur according to a given point process, and removals occur at the left edge, with their number up to time t denoted by J_t^N. The space-time injection and removal locations are denoted by INJ_i^N and REM_i^N, $i \in \mathbb{N}$, respectively, and are encoded in random measures on $\mathbb{R} \times \mathbb{R}_+$, namely

$$\alpha^N(dx, dt) = \sum_{i \in \mathbb{N}} \delta_{\mathrm{INJ}_i^N}(dx, dt), \qquad \beta^N(dx, dt) = \sum_{i \in \mathbb{N}} \delta_{\mathrm{REM}_i^N}(dx, dt). \tag{1.3}$$

Our scaling assumption is that $(\bar{\xi}_0^N, \bar{\alpha}^N, \bar{J}^N) \to (\xi_0, \alpha, J)$ in probability, where the latter is a deterministic tuple, and J is absolutely continuous, nondecreasing and null at zero (recall that the bar notation stands for normalization by N).

We can now provide a formal derivation of a PDE formulation that does not involve a free boundary. The fact that the removal mechanism acts on the leftmost particle of the configuration can be expressed as a condition on $(\bar{\xi}^N, \bar{\beta}^N)$, namely

$$\bar{\beta}^N\big(\{(x,t) \in \mathbb{R} \times \mathbb{R}_+ : \bar{\xi}_t^N(-\infty, x) > 0\}\big) = 0. \tag{1.4}$$

A key point is that $\bar{\beta}^N$ plays an additional role in the model, namely it drives the dynamics. Let us assume that in some sense $(\bar{\xi}^N, \bar{\beta}^N) \to (\xi, \beta)$ as $N \to \infty$, and moreover, that ξ_t has a density $u(\cdot, t)$ for each t. Then the macroscopic dynamics should satisfy

$$\partial_t u - \frac{1}{2}\partial_x^2 u - \kappa u = \alpha - \beta.$$

Thus, one is led to the following problem formulation. Let data (ξ_0, α, J) be given. Denote $I_t = \alpha(\mathbb{R} \times [0, t])$. Assume that the macroscopic total mass remains positive, namely that if $m_t = 1 + \kappa \int_0^t m_s\, ds + I_t - J_t$ then $m_t > 0$ for all t. Find (u, β), u nonnegative, such that

$$\begin{cases} \partial_t u - \frac{1}{2}\partial_x^2 u - \kappa u = \alpha - \beta, \\ \beta(U > 0) = 0 \quad \text{where } U(x,t) = \int_{-\infty}^x u(y,t)\, dy, \\ \beta\big(\mathbb{R} \times [0,t]\big) = J_t, \\ u(\cdot, 0) = \xi_0. \end{cases} \tag{1.5}$$

The precise definition of solutions to a second-order parabolic equation with measure-valued right-hand side and measure initial condition is given in Section 2. We will refer to this as the *weak FBP formulation*, and to the condition $\beta(U > 0) = 0$ as the *complementarity condition*.

Our main result, Theorem 2.5, states that, under mild assumptions on α and J, there exists a unique solution (u, β) to (1.5) and, moreover, $(\bar{\xi}^N, \bar{\beta}^N) \to (\xi, \beta)$ in probability, where $\xi_t(dx) = u(x,t)\, dx$. The result is stated in a broader set up in which the particles follow a diffusion process on the line. To recapitulate, this formulation circumvents the nontrivial obstacle of determining conditions for existence of a free boundary as a regular trajectory and related convergence issues and, moreover, makes it possible to argue via PDE uniqueness.

1.3. *Related work.* *Particle systems with selection and related models*. A model for motionless nonlocally branching particles with selection was studied in [16], and its HDL was proved to be given in terms of an integrodifferential FBP. HDL for a model that involves injection and selection was studied in [9], where particles perform random walks on $[0, N] \cap \mathbb{Z}$. A variant of the N-BBM, in which branching is nonlocal, was studied in [13], where the HDL was proved to exist with explicit bounds on the rates. The characterization of the limit as the solution of a FBP was also proved conditionally on existence of a classical solution to the latter, but existence is not known in general. N-BBM in higher dimension with a radially symmetric fitness function was studied in [2, 3]. Recently, in [20], the HDL of a system of Brownian particles with selection was characterized via the inverse first-passage time problem.

There are both formal and rigorous relations between FBP (1.1) and (1.2) and the Stefan FBP [2, 10]. The latter was obtained as limits of variants of the symmetric simple exclusion process (SSEP) in [21, 22], as well as the limit of interacting diffusions with rank-dependent drift in [8]. In [11], a SSEP with birth of the leftmost hole and death of the rightmost particle was considered, and convergence at the hydrodynamic scale was proved; the question of a rigorous connection to a FBP was left open.

In addition to proving HDL, some of the aforementioned papers have studied the long time behavior of the macroscopic dynamics and the interchange of the N and the t limits [2, 3, 10–12, 16].

On earlier weak formulations. Relaxed solutions to (1.2) were proposed in [10], defined by the limit of a sequence of classical solutions to FBP with perturbed data, as the perturbation size tends to zero. The existence of these classical solutions was proved via local existence to the Stefan problem with piecewise C^1 free boundary. To the best of our knowledge, this idea has been implemented only for the model studied in [10], where injection and removal rates are constant. The paper [14] introduced a probabilistic reformulation of the Stefan FBP and used it to define solutions beyond singularities, known to occur in the supercooled case of the problem. While the equations are related, this formulation does not directly apply to the FBP considered here.

On the barrier method. The use of barriers was introduced into the subject in [9] and [11] (for particle systems with topological interaction, and, respectively, SSEP with free boundaries). In this context, barriers are discrete versions of the macroscopic dynamics defined by a Trotter procedure, that bound below and above solutions to a FBP. They have been used to obtain uniqueness by showing that there can be at most one element separating all lower barriers from all upper barriers. Similar ideas have been used at the level of stochastic particle systems to provide a.s. bounds on particle system dynamics. The use of deterministic barriers to proving uniqueness of a relaxed FBP solution first appeared in [10]. Variants of the method have since been used in several papers including [4, 12, 13].

A key to proving that barriers form bounds on FBP solutions is a Feynman–Kac representation. This tool is missing in the generality at which (1.5) is considered in this paper, for reasons having to do with the irregularity of the free boundary. Consequently, the structure of, and argument behind, the (lower) barriers developed here differs from that in the above references. This issue is discussed in Remark 2.6 and Section 3.3.

1.4. *Organization of the paper.* In Section 2, the injection-branching-selection model is constructed, the weak FBP formulation is defined, and the main result, Theorem 2.5, is stated. The remaining sections provide the proofs. In Section 3, the barrier method is used to prove PDE uniqueness, starting from properties of mild solutions in Section 3.1, and properties of operators required for the construction of the barriers in Section 3.2. In Section 3.3, the construction of barriers in earlier work is described, and a sketch of their use in this paper is provided. The barriers are constructed in Section 3.4 and Section 3.5. The proof of uniqueness is completed in Section 3.6. The convergence is proved in Section 4, where Section 4.1 studies the measure-valued prelimit process (Lemma 4.1), establishes tightness of $(\bar{\xi}^N, \bar{\beta}^N)$ (Lemma 4.2) and measurability of limit densities (Lemma 4.3). Section 4.2 shows that the complementarity condition is preserved under the limit (Lemma 4.4), and finally, Section 4.3 completes the proof of the main result.

1.5. *Notation.* Denote by $\mathbb{N}$ (respectively, $\mathbb{Z}_+$) the set of positive (respectively, nonnegative) integers. For $N \in \mathbb{N}$, $[N] := \{1, 2, \dots, N\}$. Let $\mathbb{B}_r(x) = \{y \in \mathbb{R} : \|y - x\| \le r\}$. Denote by $\mathcal{M}(\mathbb{R})$ the space of finite signed Borel measures on $\mathbb{R}$ endowed with the topology of weak convergence. Let $\mathcal{M}_1(\mathbb{R}) \subset \mathcal{M}_+(\mathbb{R}) \subset \mathcal{M}(\mathbb{R})$ denote the subsets of probability, and respectively positive measures, and give them the inherited topologies. Denote $\mathbb{R}_+ = [0, \infty)$ and let $\mathcal{M}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}_+)$ be the space of signed Borel measures on $\mathbb{R} \times \mathbb{R}_+$ that are finite on $\mathbb{R} \times [0, T]$ for every T and give it the topology of weak convergence on $\mathbb{R} \times [0, T]$ for every T. Similarly, let $\mathcal{M}_{+,\mathrm{loc}}(\mathbb{R} \times \mathbb{R}_+) \subset \mathcal{M}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}_+)$ be the subspace of positive measures with inherited topology. For $X = \mathbb{R}$ or $\mathbb{R} \times \mathbb{R}_+$, for $\mu \in \mathcal{M}_+(X)$, denote the total mass by $|\mu| = \mu(X)$ and the support by supp μ. For $\mu, \nu \in \mathcal{M}_+(X)$, write $\mu \sqsubset \nu$ if $\mu(A) \le \nu(A)$ for all measurable $A \subset X$.

For $u, v \in \mathcal{B}(\mathbb{R}, \mathbb{R})$ (Borel measurable) and $\xi \in \mathcal{M}_+(\mathbb{R})$, denote $\langle u, \xi \rangle = \int u \, d\xi$ and $(u, v) = \int uv \, dx$. For $\xi \in \mathcal{M}_+(\mathbb{R})$, $u \in L_1(\mathbb{R})$ and an interval, say $[a, b]$, use $\xi[a, b]$ and $u[a, b]$ as shorthand for $\xi([a, b])$ and, respectively, $\int_a^b u \, dx$.

For $p \in [1, \infty]$, abbreviate $L_p(\mathbb{R})$ to L_p, and for $u \in L_p$ denote $\|u\|_p = \|u\|_{L_p}$. For $p, q \in [0, \infty]$, let $L_{p,\mathrm{loc}}(\mathbb{R}_+, L_q)$ denote the linear space of functions from $\mathbb{R}_+$ to L_q that are p-integrable on $[0, T]$ for every T, that is, $\int_0^T \|u(\cdot, t)\|_q^p \, dt < \infty$ if $p \in [1, \infty)$ and $\operatorname{ess\,sup}\{\|u(\cdot, t)\|_q : t \in [0, T]\} < \infty$ if $p = \infty$, equipped with the corresponding norm. Define $L_{p,\mathrm{loc}}((0, \infty), L_q)$ similarly, with $[0, T]$ replaced by $[T_1, T_2]$, $0 < T_1 < T_2 < \infty$. A member $u = u(x, t)$ of $L_{p,\mathrm{loc}}(\mathbb{R}_+, L_q)$ is said to be a.e. nonnegative if, for a.e. t, it is nonnegative for a.e. x.

For (X, d_X), a Polish space let $C(\mathbb{R}_+, X)$ and $D(\mathbb{R}_+, X)$ denote the space of continuous and respectively càdlàg paths, endowed with the topology of uniform convergence on compacts and respectively the Skorohod J_1 topology. Let $C^\uparrow(\mathbb{R}_+, \mathbb{R}_+)$ denote the subset of $C(\mathbb{R}_+, \mathbb{R}_+)$ of nondecreasing functions that vanish at zero. For $I \in C^\uparrow(\mathbb{R}_+, \mathbb{R}_+)$, denote by dI_t the corresponding Stieltjes measure on $\mathbb{R}_+$. Denote by $AC^\uparrow(\mathbb{R}_+, \mathbb{R}_+)$ the subset of $C^\uparrow(\mathbb{R}_+, \mathbb{R}_+)$ of absolutely continuous functions. For $\rho \in (0, 1]$, let $C^\rho(\mathbb{R}_+, \mathbb{R})$ denote the space of ρ-Hölder continuous functions starting at zero. Denote by $C_c^\infty(X)$ the space of compactly supported smooth functions on X when $X = \mathbb{R}$ or $\mathbb{R} \times \mathbb{R}_+$. For $f : \mathbb{R}_+ \to X$, denote

$$w_{[T_1,T_2]}(f, \delta) = \sup\{d_X(f(s), f(t)) : T_1 \le s \le t \le (s + \delta) \wedge T_2\},$$

and $w_T = w_{[0,T]}$. For $(Y, |\cdot|)$ a normed space and $f : \mathbb{R}_+ \to Y$, denote

$$\|f\|^*_{[T_1,T_2]} = \sup\{|f(s)| : t \in [T_1, T_2]\} \quad \text{and} \quad \|f\|^*_T = \|f\|^*_{[0,T]}.$$

The term *with high probability* (w.h.p.) means "away from an N-dependent event whose probability tends to zero as $N \to \infty$." The symbol c denotes a positive constant whose value may change from one expression to another.

2. Particle system model, weak formulation and main result.

2.1. *Particle system construction.* First, we describe the motion that individual particles perform, namely a diffusion process with coefficients $\mathfrak{b}$ and $\mathfrak{c}$ satisfying the following.

ASSUMPTION 2.1. One has $\mathfrak{b} \in C^1(\mathbb{R})$ and $\mathfrak{c} \in C^2(\mathbb{R})$ with $\mathfrak{b}$, $\mathfrak{c}$ and its derivative $\mathfrak{c}'$ bounded and $\mathfrak{c}$ bounded away from zero.

Given a one-dimensional BM B, we will denote by $\mathfrak{X}(x, s, B)$ the unique strong solution $\{X_t : t \in [s, \infty)\}$ to the SDE

$$X_t = x + \int_s^t \mathfrak{b}(X_\theta) \, d\theta + \int_s^t \mathfrak{c}(X_\theta) \, dB_\theta, \quad t \in [s, \infty). \tag{2.1}$$

The particle system, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is indexed by N, where N is the initial number of particles. The particles are indexed by the set $\mathcal{S} = \mathbb{N} \times \mathbb{Z}_+$, where particles $i = (j, 0)$ are roots of family trees, and particles $i = (j, k)$, $k \geq 1$ are descendants of $(j, 0)$. Below, items (S.1)–(S.5) list stochastic primitives of the model, and (S.6) states a condition they satisfy.

S.1 A collection $\{x^i(N) : i = (j, 0),\ j \in [N]\}$ of real-valued random variables representing the initial positions of the particles in the initial configuration. For such i set $\sigma^i(N) = 0$, expressing the fact that these particles are present in the system at time 0.
S.2 A collection $\{(x^i(N), \sigma^i(N)) : i = (j, 0),\ j = N+1, N+2, \ldots\}$ of $\mathbb{R} \times (0, \infty)$-valued random variables representing the initial space-time positions of injected particles, ordered by injection time, assumed distinct: $\sigma^{(N+1,0)}(N) < \sigma^{(N+2,0)}(N) < \cdots$.
S.3 A collection $\{B^i : i \in \mathcal{S}\}$ of mutually independent BM, driving the motion of the corresponding particles.
S.4 A collection $\{\pi^i : i \in \mathcal{S}\}$ of mutually independent rate-κ Poisson processes, where $\kappa \geq 0$ is the branching rate, determining the times a living particle gives birth.
S.5 A sequence $0 < \eta^1(N) < \eta^2(N) < \cdots$ of removal attempt times.
S.6 The first four stochastic elements (S.1)–(S.4) are mutually independent.

The notation $x^i(N)$, $\sigma^i(N)$ and $\eta^l(N)$ is henceforth abbreviated to x^i, σ^i and η^l.

The construction based on these primitives in presented momentarily, but first it is helpful to clarify two points. First, the reason η^l are called removal attempt times, not removal times, is that it is possible that there are no particles in the system when one of these times occurs, in which case no particle is actually removed (see the 4th bullet below). Second, by the independence assumption, a.s., no simultaneous introduction of particles (by injection or birth) can occur after time 0. However, the introduction of a new particle and the removal of a particle may occur simultaneously, such as when branching and removals are coupled (see the last bullet below).

The term *leftmost particle* refers to the particle whose position is lowest among the particles in the configuration at a given time. Ties are broken according to some fixed ordering of the labels. A particle is said to be *introduced* into the system at a certain time if it is either injected (as in (S.2)) or born out of a branching event (as is (S.4)).

The construction now proceeds in two steps. First, once the initial space-time position (x^i, σ^i) of particle i is determined, its *potential trajectory*, denoted $\{X^i_t, t \in [\sigma^i, \infty)\}$, is defined by

$$X^i_t = \mathfrak{X}(x^i, \sigma^i, B^i)(t), \quad t \geq \sigma^i. \tag{2.2}$$

In the second step, the removal time τ^i of particle i is determined (where ∞ is possible), and the *actual trajectory* the particle follows is obtained by trimming the potential trajectory at τ^i.

The particle configuration is defined on $(\eta^l, \eta^{l+1}]$ inductively for $l = 0, 1, 2, \ldots$, where $\eta^0 = 0$. The configuration at time 0 is given by $X^i_0 = x^i$ for $i = (j, 0)$, $j \in [N]$. This gives a well-defined potential trajectory of each of these particles on $[0, \infty)$. Next, for $l \geq 0$, given the configuration during $[0, \eta^l]$, the construction during $(\eta^l, \eta^{l+1}]$ is described as follows.

During the time interval (η^l, η^{l+1}):

- Each of the particles living at η^l already has a well-defined potential trajectory. These particles live through the interval, with their actual trajectories given by their potential trajectories.
- Each i of the form $(j, 0)$ with $\sigma^i \in (\eta^l, \eta^{l+1})$ corresponds to an injection during this interval. This determines the injection space-time location (x^i, σ^i) of a new particle, and accordingly its potential trajectory for all $t \geq \sigma^i$. These particles live through the remainder interval.

- If a particle $i = (j, k)$ is alive when π^i ticks, it gives birth to a new particle at that space-time location. The new particle gets the label $\hat{i} = (j, \hat{k}+1)$ where $(j, \hat{k})$ is the latest descendant of $(j, 0)$ introduced prior to that time. Again, this determines the potential trajectory, and the particle lives through the remainder interval.

At the time η^{l+1}:

- If there are no particles in the system (i.e., the configuration at time η^l is empty), nothing happens. Otherwise:
- If no new particle is introduced at that time then the particle that is leftmost at $\eta^{l+1}-$ is removed. If the index of this particle is i, then this determines its removal time as $\tau^i = \eta^{l+1}$.
- If a particle is introduced at η^{l+1} (by injection or branching), the construction obeys the rule "introduce and then remove," and this may cause the new particle to be removed immediately (i.e., if the injection is to the left of all particles or the branching particle is the leftmost).

For particles i that never get removed, define $\tau^i = \infty$. The lifetime of particle i is given by $[\sigma^i, \tau^i)$ (regarded empty if $\sigma^i = \tau^i$ or $\sigma^i = \infty$) and its actual trajectory is defined by $\{X_t^i : t \in [\sigma^i, \tau^i)\}$. This completes the construction of the particle system.

Some useful notation is as follows. The set of particles initially in the system, injected and respectively descendants of a root particle $i = (j, 0)$, are denoted by

$$\mathcal{S}^{N,\text{init}} = [N] \times \{0\}, \qquad \mathcal{S}^{N,\text{inj}} = \{N+1, N+2, \dots\} \times \{0\}, \qquad \mathcal{S}^{N,i} = \{(j,k) : k \in \mathbb{Z}_+\}.$$

The set of particles introduced by time t is

$$\mathcal{S}_t^N = \{i \in \mathcal{S} : \sigma^i \le t\}.$$

Those injected by time t and respectively descendants of root particle i introduced by time t, are denoted by

$$\mathcal{S}_t^{N,\text{inj}} = \mathcal{S}^{N,\text{inj}} \cap \mathcal{S}_t^N, \qquad \mathcal{S}_t^{N,i} = \mathcal{S}^{N,i} \cap \mathcal{S}_t^N. \tag{2.3}$$

Next, the configuration process is given by

$$\xi_t^N(dx) = \sum_{i \in \mathcal{S}} \delta_{X_t^i}(dx) 1_{\{\sigma^i \le t < \tau^i\}},$$

and clearly its initial condition is

$$\xi_0^N(dx) = \sum_{i \in \mathcal{S}^{N,\text{init}}} \delta_{x^i}(dx).$$

The injection and respectively removal space-time locations are encoded by the random measures

$$\alpha^N(dx, dt) = \sum_{i \in \mathcal{S}^{N,\text{inj}}} \delta_{(x^i, \sigma^i)}(dx, dt), \qquad \beta^N(dx, dt) = \sum_{i \in \mathcal{S}: \tau^i < \infty} \delta_{(X_{\tau^i}^i, \tau^i)}(dx, dt). \tag{2.4}$$

Let $m_t^N = |\xi_t^N|$ denote the number of living particles at t. Let $I_t^N = \#\mathcal{S}_t^{N,\text{inj}}$ be the number of injections by time t. Then $I_t^N = \alpha^N(\mathbb{R} \times [0, t])$. Moreover, let $J_t^N = \#\{l \ge 1 : \eta^l \le t\}$ be the number of removal attempts by t. Note that the actual number of removals by t is $\beta^N(\mathbb{R} \times [0, t])$. Then $J_t^N = \beta^N(\mathbb{R} \times [0, t])$ holds on the event $\{\inf_{s \le t} m_s^N \ge 1\}$.

So far we have not made any assumption on the removal attempt times η^l. We would like to cover the possibility that removals are coupled with (some of the) injections or branching events, as well as that they occur independently of each other. Hence we let

$$\mathcal{F}_t^N = \sigma\{\xi_0^N, \alpha^N(-\infty, x] \times [0, s], B_s^i, \pi_s^i, J_s^N : x \in \mathbb{R}, s \in [0, t], i \in \mathcal{S}\}, \tag{2.5}$$

and supplement (S.1)–(S.6) above with:

S.7 B^i and $\hat{\pi}^i$ are $\{\mathcal{F}_t^N\}$-martingales, $i \in \mathcal{S}$, where $\hat{\pi}^i(t) = \pi^i(t) - \kappa t$.

2.2. *Macroscopic data.*

DEFINITION 2.2. An *admissible macroscopic data* is a deterministic tuple (ξ_0, α, J) satisfying the following conditions, for some $\rho_0 > 0$:

(i) $\xi_0 \in \mathcal{M}_1(\mathbb{R})$, $\alpha \in \mathcal{M}_{+,\mathrm{loc}}(\mathbb{R} \times \mathbb{R}_+)$ and $J \in AC^\uparrow(\mathbb{R}_+, \mathbb{R}_+) \cap C^{\rho_0}(\mathbb{R}_+, \mathbb{R}_+)$.

(ii) Denote $I_t = \alpha(\mathbb{R} \times [0, t])$. Then one of the following holds:

1. $\alpha(dx, dt) \sqsubset c\, dx\, dI_t$, some $c < \infty$. Moreover, $I \in C^\uparrow(\mathbb{R}_+, \mathbb{R}_+) \cap C^{\rho_0}(\mathbb{R}_+, \mathbb{R}_+)$.
2. $I \in C^\uparrow(\mathbb{R}_+, \mathbb{R}_+) \cap C^{\frac{1}{2}+\rho_0}(\mathbb{R}_+, \mathbb{R}_+)$.

(iii) Let m denote the solution to

$$m_t = 1 + \kappa \int_0^t m_s\, ds + I_t - J_t, \tag{2.6}$$

representing the total macroscopic mass. Then $\varepsilon_0 := \inf_{t \in \mathbb{R}_+} m_t > 0$.

The stochastic elements α^N and β^N (resp., ξ_0^N; ξ^N; I^N and J^N) are viewed as random variables taking values in $\mathcal{M}_{+,\mathrm{loc}}(\mathbb{R} \times \mathbb{R}_+)$ (resp., $\mathcal{M}_1(\mathbb{R})$; $D(\mathbb{R}_+, \mathcal{M}_+(\mathbb{R}))$; $D(\mathbb{R}_+, \mathbb{R})$). The assumed structure of the stochastic primitives is as follows (recall that the bar notation stands for normalization by N).

ASSUMPTION 2.3. As $N \to \infty$, $(\bar{\xi}_0^N, \bar{\alpha}^N, \bar{J}^N) \to (\xi_0, \alpha, J)$ in the product topology, in probability, where the latter is an admissible macroscopic data.

REMARK 2.4. (i) (N-BBM as a special case). In the N-BBM model, branching occurs at rate 1 per particle, and removal and branching are coupled, so that the number of particles remains N at all times. Thus, the birth/removal counting process, J^N, is Poisson of rate N. This model satisfies our assumptions: Assumption 2.3 is satisfied with $\alpha = 0$, $\kappa = 1$, $J_t = t$ and $(\mathfrak{b}, \mathfrak{c}) = (0, 1)$, while condition (S.7) can be seen to hold under the aforementioned coupling. Next, if one abandons the requirement that births and removals are coupled, and instead assumes that the removal count J^N and the branching mechanism are mutually independent, but J^N is still Poisson of rate N, then all assumptions are still valid with the same macroscopic data. This latter version can be extended to allow variable removal rate by letting $J \in AC^\uparrow(\mathbb{R}_+, \mathbb{R}_+)$ and assuming that m_t defined by

$$m_t = 1 + \int_0^t m_s\, ds - J_t$$

remains positive at all times. Letting J^N be an inhomogeneous Poisson process with instantaneous intensity $N J_t'$ results in a BBM with macroscopic mass m_t.

(ii) (The model from [10]). Similarly, the injection-selection model of [10], mentioned in the Introduction, is closely related. In this model, the injections and removals are again coupled. Consider $\kappa = 0$, α^N a Poisson point process with intensity $N\pi(dx)c_0\, dt$, where π is any probability measure and $c_0 > 0$ a constant, and $J^N = I^N = \alpha^N(\mathbb{R} \times [0, t])$. Then the case $\pi = \delta_0$ gives the model from [10] except the minor point that the particles live in $\mathbb{R}$ rather than $\mathbb{R}_+$.

2.3. *Weak FBP formulation and main result.* First, we recall the notion of second-order parabolic equations with a measure-valued right-hand side. Let $\mathfrak{a} = \frac{1}{2}\mathfrak{c}^2$ and

$$\mathcal{L}\varphi = \mathfrak{a}\partial_x^2\varphi + \mathfrak{b}\partial_x\varphi + \kappa\varphi, \qquad \mathcal{L}^* u = \partial_x^2(\mathfrak{a}u) - \partial_x(\mathfrak{b}u) + \kappa u. \tag{2.7}$$

For $\xi_0 \in \mathcal{M}_1(\mathbb{R})$, $\mu \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}_+)$, consider the equation

$$\begin{cases} \partial_t u - \mathcal{L}^* u = \mu, \\ u(\cdot, 0) = \xi_0. \end{cases} \tag{2.8}$$

Let $q \in (1, \infty)$. A weak L_q-solution of (2.8) is a function $u \in L_{1,\mathrm{loc}}(\mathbb{R}_+, L_q)$ satisfying

$$-\int_0^\infty (\partial_t \varphi + \mathcal{L}\varphi, u)\, dt = \int_{\mathbb{R}} \varphi(\cdot, 0)\, d\xi_0 + \int_{\mathbb{R}\times\mathbb{R}_+} \varphi\, d\mu \tag{2.9}$$

for all $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+)$.

Such problems were analyzed in [1]. In particular, [1], Theorems 1 and 13 and Remarks 1 and 2(b), show that for $1 < q < \infty$, this problem possesses a unique weak L_q-solution, independent of q (note that with the transformation $\bar{\mathfrak{a}} = \mathfrak{a}$, $\bar{\mathfrak{b}} = -\mathfrak{b} + \mathfrak{a}'$, $\bar{\kappa} = \kappa - \mathfrak{b}'$, one has the divergence form $\mathcal{L}^* u = \partial_x(\bar{\mathfrak{a}}\partial_x u) + \bar{\mathfrak{b}}\partial_x u + \bar{\kappa} u$, as required in [1], Remark 1(e); also note that in [1] the initial condition is accounted for by substituting $\mu + \xi_0 \otimes \delta_0$ for μ). In what follows, we thus use the term weak solution to (2.8), without reference to q.

We base on this notion the following problem formulation. Let admissible data (ξ_0, α, J) be given. Consider the equation

$$\begin{cases} \text{(i)} \quad \partial_t u - \mathcal{L}^* u = \alpha - \beta, \\ \text{(ii)} \quad \beta(U > 0) = 0 \qquad \text{where } U(x,t) = \int_{-\infty}^x u(y,t)\, dy, \\ \text{(iii)} \quad \beta\big(\mathbb{R} \times [0,t]\big) = J_t \qquad \text{for } t \in \mathbb{R}_+, \\ \text{(iv)} \quad u(\cdot, 0) = \xi_0. \end{cases} \tag{2.10}$$

A solution (u, β) to (2.10) is defined as a member of $L_{1,\mathrm{loc}}(\mathbb{R}_+, L_q) \times \mathcal{M}_{+,\mathrm{loc}}(\mathbb{R} \times \mathbb{R}_+)$ for some (equivalently, all) $q \in (1, \infty)$, such that u is an a.e. nonnegative weak solution to (2.10)(i, iv) and, moreover, conditions (2.10)(ii, iii) hold.

For future reference, according to (2.9), a weak solution to (2.10)(i, iv) is one for which

$$-\int_0^\infty (\partial_t\varphi + \mathcal{L}\varphi, u)\, dt = \int_{\mathbb{R}} \varphi(\cdot, 0)\, d\xi_0 + \int_{\mathbb{R}\times\mathbb{R}_+} \varphi\, d\alpha - \int_{\mathbb{R}\times\mathbb{R}_+} \varphi\, d\beta, \tag{2.11}$$

for $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+)$.

THEOREM 2.5. *Let Assumptions 2.1 and 2.3 hold. Then*:

(i) *There exists a unique solution (u, β) to (2.10).*

(ii) *There exists a version of u, again denoted u, such that setting $\xi_t(dx) = u(x,t)\, dx$ gives $\xi \in C(\mathbb{R}_+, \mathcal{M}_+(\mathbb{R}))$, and $(\bar{\xi}^N, \bar{\beta}^N) \to (\xi, \beta)$ in $D(\mathbb{R}_+, \mathcal{M}_+(\mathbb{R})) \times \mathcal{M}_{+,\mathrm{loc}}(\mathbb{R} \times \mathbb{R}_+)$, in probability, as $N \to \infty$.*

REMARK 2.6. (i) The notion (2.10) can be seen as an extension of a classical solution to a FBP. For example, consider (1.2) and assume that (u, ℓ) is a classical solution, with $\ell \in C(\mathbb{R}_+, \mathbb{R})$ and u appropriately smooth. Recall that the injection-removal rate is set to c_0 in (2.10). Then a solution (u, β) to (2.10) is obtained by $\beta(dx, dt) = \delta_{\ell_t}(dx) c_0\, dt$, as can be verified directly.

(ii) More generally, one can construct a free boundary out of any solution (u, β) of (2.10). To this end, one can take, as in Theorem 2.5, $\xi_t(dx) = u(x,t)\, dx$, and disintegrate β in the form $\beta(dx, dt) = \beta_t(dx)\, dJ_t$. Then two candidates are given by

$$\ell_t = \inf \operatorname{supp} \xi_t, \qquad \tilde{\ell}_t = \sup \operatorname{supp} \beta_t, \quad t \in (0, \infty).$$

These may in general behave very irregularly. For example, consider $\mathcal{L}^*u = \partial_x^2 u + u$ and $\alpha = 0$, and for $K \in \mathcal{B}(\mathbb{R}_+)$, let $J_t = |[0,t] \cap K|$. (Note that J_t satisfies our assumptions, and thus ξ, β, ℓ and $\tilde{\ell}$ are well-defined.) Now, if $K = [0,1] \cup [2,\infty)$ then during the time interval $(0,1)$, the free boundary, say ℓ, evolves continuously (by results of [4]), at time 1 it jumps to $-\infty$ (as there is no absorption of mass), and at time 2 it comes back from $-\infty$. It is clear that this behavior can be made far more complicated by picking other choices of K, and for a general Borel set K, one does not expect any regularity of ℓ beyond Borel measurability.

Under such circumstances, the notion of a free boundary does not seem to be particularly useful from an analytic point of view (e.g., since boundary conditions in initial boundary value problems are typically specified on the boundary of an open set). Also, a Feynman–Kac representation of the component u of the solution in terms of ℓ, which involves the hitting time of BM to ℓ, requires some regularity of ℓ (e.g., piecewise continuity). Feynman–Kac representation of u in terms of ℓ has served as a key tool in earlier work on the subject (see details in Section 3.3), but for the reasons mentioned, is not available here.

3. Uniqueness via barriers.

In this section, we prove the following result.

THEOREM 3.1. *Let Assumption 2.1 hold and let admissible data* (ξ_0, α, J) *be given. Then there exists at most one solution* (u, β) *to* (2.10).

The proof is based on the construction of barriers, which are shown to constitute upper and lower bounds to any solution, in the sense of mass transport inequalities. This section is structured as follows. Essential tools are developed in Section 3.1 and Section 3.2, where the former provides so-called mild solutions, and the latter introduces operators required for the construction, and studies some of their properties. Section 3.3 gives a brief description of the construction of barriers in earlier work and of the difference to that in the current work, as well as a sketch of the proof of uniqueness based on them. The upper and lower barriers are constructed in Section 3.4 and Section 3.5, respectively. In Section 3.6, it is shown that the barriers can be made close to each other, and the proof is completed.

3.1. *Preliminary lemmas.*

The backward Kolmogorov equation associated with the diffusion (2.1) is given by $\partial_t u = \mathcal{L}_1 u$ with $\mathcal{L}_1 u = \mathfrak{a}\partial_x^2 u + \mathfrak{b}\partial_x u$. Denote by $\mathfrak{p}_t(x,y)$ the fundamental solution of this equation.

LEMMA 3.2. *Given* T *there exist constants* $\tilde{c}_1, \tilde{c}_2, \hat{c}_1, \hat{c}_2 > 0$ *such that for* $t \in (0,T]$ *and* $x, y \in \mathbb{R}$,

$$\tilde{c}_1 t^{-1/2} e^{-\tilde{c}_2 (x-y)^2 t^{-1}} \le \mathfrak{p}_t(x,y) \le \hat{c}_1 t^{-1/2} e^{-\hat{c}_2 (x-y)^2 t^{-1}}. \tag{3.1}$$

Whereas the upper bound will be used many times, the lower bound is needed only to make the following statement (used in Lemma 3.8). There exists a constant $c_* > 0$ such that

$$\int_{-\infty}^{x} \mathfrak{p}_t(x,y)\,dy \ge c_*, \qquad t \in (0,1], x \in \mathbb{R}. \tag{3.2}$$

PROOF. For $T = 1$, these bounds follow from [24], Theorems 4.4.6 and 4.4.12. To verify the assumptions, note that one can write $\mathcal{L}_1$ in the form $\mathcal{L}_1 u = \partial_x(\mathfrak{a}\partial_x u) + \tilde{\mathfrak{b}}\mathfrak{a}\partial_x u$ by setting $\tilde{\mathfrak{b}} = (\mathfrak{b} - \mathfrak{a}')\mathfrak{a}^{-1}$. The boundedness of $\tilde{\mathfrak{b}}$ follows from the assumed boundedness of $\mathfrak{a}^{-1}$, $\mathfrak{a}'$ and $\mathfrak{b}$. For $T > 1$, apply the scaling property of $\mathfrak{p}$ as in [24], Remark 4.1.5 (with constants depending on T). □

Denote

$$\mathfrak{s}_t(x,y) = e^{\kappa t}\mathfrak{p}_t(x,y). \tag{3.3}$$

For $u \in L_1(\mathbb{R})$, denote

$$S_t u(y) = \int_{\mathbb{R}} \mathfrak{s}_t(x,y)u(x)\,dx, \tag{3.4}$$

and with a slight abuse of notation use the same symbol for $\xi \in \mathcal{M}_+(\mathbb{R})$, namely

$$S_t\xi(y) = \int_{\mathbb{R}} \mathfrak{s}_t(x,y)\xi(dx). \tag{3.5}$$

For $\gamma \in \mathcal{M}_{+,\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+)$ and $0 \le \tau < t$, denote

$$S*\gamma(y,t) = \int_{\mathbb{R}\times[0,t]} \mathfrak{s}_{t-s}(x,y)\gamma(dx,ds),$$

$$S*\gamma(y,t;\tau) = \int_{\mathbb{R}\times[\tau,t]} \mathfrak{s}_{t-s}(x,y)\gamma(dx,ds).$$

LEMMA 3.3. (i) *Let $\gamma \in \mathcal{M}_{+,\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+)$ be such that $\gamma(\mathbb{R}\times[0,\cdot]) \in C^{\rho_0}(\mathbb{R}_+,\mathbb{R}_+)$ for some $\rho_0 > 0$. Let $v(y,t) = S_t\xi_0(y) + S*\gamma(y,t)$. Then, for $q \in (1,\infty)$, $v \in L_{1,\mathrm{loc}}(\mathbb{R}_+, L_q)$.*

(ii) *Let $(\tilde{u},\beta)$ be a solution to (2.10). If u is a version of $\tilde{u}$, then (u,β) is also a solution. Moreover, $\tilde{u}$ has a version given by*

$$u(y,t) = S_t\xi_0(y) + S*\alpha(y,t) - S*\beta(y,t). \tag{3.6}$$

(iii) *One has $\|u(\cdot,t)\|_1 = m_t$, $t>0$. Moreover, $v,u \in L_{\infty,\mathrm{loc}}((0,\infty),L_\infty)$.*

(iv) *One has, for $0<\tau<t$,*

$$u(y,t) = S_{t-\tau}u(\cdot,\tau)(y) + S*\alpha(y,t;\tau) - S*\beta(y,t;\tau). \tag{3.7}$$

REMARK 3.4. In what follows, by a solution to (2.10) we will mean the version given by (3.6) unless stated otherwise. In view of Lemma 3.3, there is no loss of generality in doing so when proving the uniqueness result.

PROOF. (i) Fix T. In this proof, c denotes a constant not depending on x, y and $t \in (0,T]$ whose value may change from one expression to another. By Lemma 3.2 and (3.3), it is easy to see that $\|\mathfrak{s}_t(x,\cdot)\|_2 \le ct^{-1/2}$, $t\in(0,T]$. By Minkowski's integral inequality, it follows that $\|S_t\xi_0\|_2 \le ct^{-1/2}$.

Next, let $q\in(1,\infty)$. Then, by Lemma 3.2 and (3.3), for $t\in(0,T]$, $\|\mathfrak{s}_t(x,\cdot)\|_q \le ct^{-Q}$ where $Q=(q-1)/(2q)$. By Minkowski's integral inequality,

$$\|S*\gamma(\cdot,t)\|_q \le c\int_{\mathbb{R}\times[0,t]}(t-s)^{-Q}\gamma(dx,ds) = c\int_{[0,t]}(t-s)^{-Q}\,dK_s,$$

where $K_t = \gamma(\mathbb{R}\times[0,t])$. By monotone convergence, the last integral is the limit as $\varepsilon \downarrow 0$ of

$$\begin{aligned}
&\int_{[0,t-\varepsilon]}(t-s)^{-Q}\,dK_s \\
&= K_{t-\varepsilon}\varepsilon^{-Q} - Q\int_0^{t-\varepsilon} K_s(t-s)^{-1-Q}\,ds \\
&= (K_{t-\varepsilon}-K_t)\varepsilon^{-Q} + Q\int_0^{t-\varepsilon}(K_t-K_s)(t-s)^{-1-Q}\,ds + K_t t^{-Q} \\
&\le c\int_0^t (t-s)^{\rho_0-1-Q}\,ds + K_t t^{-Q}.
\end{aligned} \tag{3.8}$$

If we choose $q > 1$ sufficiently small, then $\rho_0 - 1 - Q \in (-1, 0)$, and the above integral is bounded by c for $t \le T$. Moreover, $Q \in (0, 1)$, and we obtain that $\|S * \gamma(\cdot, t)\|_q$ is integrable over $[0, T]$.

The two terms defining v satisfy (2.9) with data $(\xi_0, 0)$ and $(0, \gamma)$, respectively, by elementary integration by parts (see, e.g., [18], Theorem 4.6, for a similar calculation). The estimates above on these two terms show that they are respectively members of $L_{1,\mathrm{loc}}(\mathbb{R}_+, L_2)$, and $L_{1,\mathrm{loc}}(\mathbb{R}_+, L_q)$ for q close to 1. Hence each is a weak L_q-solution of the corresponding equation for some $q \in (1, \infty)$. By the results of [1] discussed following (2.9), they must therefore be the unique weak L_q-solution, for all $q \in (1, \infty)$. This proves the assertion.

(ii) For the first assertion, we must show that the complementary condition (2.10)(ii) is preserved by changing $\tilde{u}$ to u. If U is as in (2.10)(ii) and $\tilde{U}$ is defined similarly, then there is a set $A \subset \mathbb{R}_+$ of full Lebesgue measure such that $U(x, t) = \tilde{U}(x, t)$ for $(x, t) \in \mathbb{R} \times A$. Owing to the assumption that J is absolutely continuous, β does not charge $\mathbb{R} \times A^c$. This shows that $\beta(U > 0)$ holds if and only if $\beta(\tilde{U} > 0)$, proving the assertion.

For the second assertion, the arguments given above in (i) show that the three terms on the right of (3.6) are, for every $q \in (1, \infty)$, weak L_q-solutions of (2.8) for $\mu = \xi_0 \otimes \delta_0$, α and respectively $-\beta$. By linearity in μ of weak solutions of (2.8) [1], Theorem 1, it follows that u defined in (3.6) is a weak L_q-solution of (2.10)(i) corresponding to data (ξ_0, α, β). In particular, u must be a version of $\tilde{u}$ by uniqueness of solutions to (2.8).

(iii) To calculate $\tilde{m}_t := \|u(\cdot, t)\|_1$, note that $\|\mathfrak{s}_t(x, \cdot)\|_1 = e^{\kappa t}$. Hence by (3.6),

$$\tilde{m}_t = e^{\kappa t} + \int_0^t e^{\kappa(t-s)}\, dI_s - \int_0^t e^{\kappa(t-s)}\, dJ_s,$$

which solves (2.6), and by uniqueness, equals m_t.

As for the estimate on $\|u(\cdot, t)\|_\infty$, by positivity we only need to estimate the first two terms in (3.6). Directly from Lemma 3.2, the sum of these two terms is bounded as follows:

$$\mathfrak{s}_t(x, y) \le ct^{-1/2} + cI_t \quad \text{and} \quad ct^{-1/2} + c\int_0^t (t-s)^{-1/2}\, dI_s,$$

under Assumption 2.3(ii.1) and respectively (ii.2). The former expression is locally bounded for t away from 0, as required. As for the latter, a calculation as in (3.8), replacing (Q, ρ_0, K_t) by $(\frac{1}{2}, \frac{1}{2} + \rho_0, I_t)$ shows that this expression is also locally bounded for t away from 0.

(iv) Finally, (3.7) follows from (3.6) upon using the Chapman–Kolmogorov equation $\int \mathfrak{p}_{t-\tau}(x, y)\mathfrak{p}_\tau(y, z)\, dy = \mathfrak{p}_t(x, z)$. $\square$

3.2. *Mass transport inequalities*. In this section, several elementary facts about mass transport inequalities are borrowed from [10], and some are developed further. On $L_1(\mathbb{R}, \mathbb{R}_+)$, define the relation $u \preccurlyeq v$ as

$$u[r, \infty) \le v[r, \infty) \quad \text{for all } r \in \mathbb{R},$$

and the relation $u \preccurlyeq v \bmod \ell$, for $\ell \ge 0$, as

$$u[r, \infty) \le v[r, \infty) + \ell \quad \text{for all } r \in \mathbb{R}.$$

For $\xi, \zeta \in \mathcal{M}_+(\mathbb{R})$, define $\xi \preccurlyeq \zeta$ and $\xi \preccurlyeq \zeta \bmod \ell$ analogously.

For $\delta > 0$, the "cut" operator C_δ acts on $H_\delta = \{u \in L_1(\mathbb{R}, \mathbb{R}_+) : \|u\|_1 > \delta\}$ by cutting mass of size $\delta > 0$ on the left. That is, for $u \in H_\delta$,

$$\Lambda_\delta(u) = \inf\{x \in \mathbb{R} : u(-\infty, x] \ge \delta\}$$

and

$$C_\delta u(x) = u(x) 1_{[\Lambda_\delta(u), \infty)}(x).$$

When $\delta = 0$ set $C_\delta = \mathtt{id}$, the identity map. Also, denote $\widehat{C}_\delta = \mathtt{id} - C_\delta$. We also use an operator that cuts out a mass of size δ lying between Λ_Δ and $\Lambda_{\Delta+\delta}$. More precisely, given $\Delta > 0$ and $\delta \ge 0$, $\hat{\delta}$ will always denote the pair (Δ, δ). Then the operator $C_{\hat{\delta}}$ acts on $H_{\Delta+\delta}$ as

$$C_{\hat{\delta}} u(x) = C_{\Delta,\delta} u(x) = u(x) 1_{\{(-\infty, \Lambda_\Delta(u)] \cup (\Lambda_{\Delta+\delta}(u), \infty)\}}(x).$$

Set $\widehat{C}_{\hat{\delta}} = \mathtt{id} - C_{\hat{\delta}}$.

LEMMA 3.5. *Let $\delta \ge 0$ and assume $u, v \in H_\delta$. Let $\ell \ge 0$.*

(i) *If $u \preccurlyeq v \bmod \ell$ and $\|u\|_1 = \|v\|_1$, then $S_\delta u \preccurlyeq S_\delta v \bmod e^{\kappa\delta}\ell$.*
(ii) *If $u \preccurlyeq v \bmod \ell$, then $C_\delta u \preccurlyeq C_\delta v \bmod \ell$.*
(iii) *If $w \in L_1(\mathbb{R}, \mathbb{R}_+)$ is such that $\|w\|_1 = \delta$ and $u - w \ge 0$, then $u - w \preccurlyeq C_\delta u$.*
Next, let $\Delta > 0$ and assume $u, v \in H_{\Delta+\delta}$.
(iv) *If $u \preccurlyeq v \bmod \ell$, then $C_{\Delta,\delta} u \preccurlyeq C_{\Delta,\delta} v \bmod \ell$.*
(v) *If $0 < \widehat{\Delta} \le \Delta$, then $C_{\Delta,\delta} u \preccurlyeq C_{\widehat{\Delta},\delta} u$.*
(vi) *If $\Delta' \ge \Delta$, $u \preccurlyeq v \bmod \Delta'$ and $\|u\|_1 = \|v\|_1$, then $C_\delta u \preccurlyeq C_{\Delta,\delta} v \bmod \Delta'$.*

PROOF. (i) Step 1. Consider the case where $\kappa = 0$ and $\|u\|_1 = \|v\|_1 = 1$. In this case, u, v are densities of probability measures. Without loss of generality (w.l.o.g.), assume $\ell < 1$. Fix r_0 be such that $\int_{r_0}^\infty u = \ell$ (where $r_0 = \infty$ if $\ell = 0$). Let $\tilde{u} = u 1_{\{\cdot \le r_0\}}$ be a density that integrates to $1 - \ell$. Consider the probability measure on $[-\infty, \infty)$, denoted $\tilde{U}$, having mass ℓ at $-\infty$ and density $\tilde{u}$ on $\mathbb{R}$. Let V be the probability measure with density v. Then one has $\tilde{U}[r, \infty) \le V[r, \infty)$ for all r. Therefore, there exists a coupling $(\tilde{X}_0, Y_0)$ having marginal distributions $\tilde{U}$ and V, respectively, such that the inequality $\tilde{X}_0 \le Y_0$ holds a.s. Denote by E the event $\{\tilde{X}_0 > -\infty\}$.

Consider a coupling of two processes $\tilde{X}$ and Y, constructed using a BM B independent of $(\tilde{X}_0, Y_0)$. Namely, on the event E, let $\tilde{X}_t$ be the unique strong solution to

$$\tilde{X}_t = \tilde{X}_0 + \int_0^t \mathfrak{b}(\tilde{X}_s)\, ds + \int_0^t \mathfrak{c}(\tilde{X}_s)\, dB_s.$$

$\tilde{X}_t$ need not be defined on E^c. Similarly, define Y_t (on all of Ω) as the solution to this SDE with initial condition Y_0. Then $\tilde{X}_t \le Y_t$ for all t holds a.s. on E. This gives

$$\mathbb{P}(\tilde{X}_\delta > r) \le \mathbb{P}\big(E \cap \{Y_\delta > r\}\big) \le \mathbb{P}(Y_\delta > r) = S_\delta v(r, \infty).$$

Next, let $X_0 = \tilde{X}_0$ on E, and let its conditional law given E^c be given by the density $\ell^{-1} u 1_{\{\cdot > r_0\}}$ (which need not be defined in the case $\ell = 0$). Then X_0 has u as its density. Again, assume w.l.o.g. that B is independent of X_0, and let X_t be defined (on all of Ω) by following the same SDE with X_0 as an initial condition. Then the density of X_δ is given by $S_\delta u$, and $X_t = \tilde{X}_t$ on E. Thus, for any $r \in \mathbb{R}$,

$$S_\delta u(r, \infty) = \mathbb{P}(X_\delta > r) \le \mathbb{P}\big(E \cap \{\tilde{X}_\delta > r\}\big) + \mathbb{P}\big(E^c\big) \le S_\delta v(r, \infty) + \ell.$$

Step 2. If $\|u\|_1 = \|v\|_1 = c$, then u/c and v/c are probability densities and $u/c \preccurlyeq v/c \bmod \ell/c$. This gives by Step 1 $S_\delta u/c \preccurlyeq S_\delta v/c \bmod \ell/c$. The claim follows on multiplying by c.

Step 3. Finally, when $\kappa > 0$, the claim follows from Step 2 after multiplying by $e^{\kappa\delta}$ and using (3.3).

(ii) Let $a = \Lambda_\delta(u)$ and $b = \Lambda_\delta(v)$. For $r \ge a \vee b$, clearly $(C_\delta u)[r, \infty) = u[r, \infty) \le v[r, \infty) + \ell = (C_\delta v)[r, \infty) + \ell$. For $r < a \vee b$, consider two cases.

Case 1: $a \le b$ and $r < b$. Then

$$(C_\delta u)[r, \infty) \le \|C_\delta u\|_1 = \|u\|_1 - \delta \le \|v\|_1 - \delta + \ell = \|C_\delta v\|_1 + \ell = C_\delta[r, \infty) + \ell.$$

Case 2: $a > b$ and $r < a$. Then

$$(C_\delta u)[r,\infty) = u[a,\infty) \le u[r \vee b,\infty) \le v[r \vee b,\infty) + \ell = (C_\delta v)[r,\infty) + \ell.$$

(iii) We have $C_\delta u = u 1_{[c,\infty)}$, where $u(-\infty, c] = \delta$. Consider $r \le c$. Because $u - w$ is nonnegative,

$$(u - w)[r,\infty) \le \|u - w\|_1 = \|u\|_1 - \|w\|_1 = \|u\|_1 - \delta = (C_\delta u)[r,\infty).$$

Next, consider $r > c$. Then $(u - w)[r,\infty) \le u[r,\infty)$ whereas $(C_\delta u)[r,\infty) = u[r,\infty)$.

(iv) We have $u[r,\infty) \le v[r,\infty) + \ell$ for all r. We must show that $\widehat{u}[r,\infty) \le \widehat{v}[r,\infty) + \ell$ for all r, where

$$\widehat{u} = u 1_{(-\infty,a)\cup(b,\infty)}, \qquad u(-\infty,a) = \Delta, \qquad u(a,b) = \delta,$$

$$\widehat{v} = v 1_{(-\infty,\bar{a})\cup(\bar{b},\infty)}, \qquad v(-\infty,\bar{a}) = \Delta, \qquad v(\bar{a},\bar{b}) = \delta.$$

We split into four cases.

Case 1. $r \le a$:

$$\widehat{u}[r,\infty) = u[r,\infty) - \delta \le v[r,\infty) - \delta + \ell \le \widehat{v}[r,\infty) + \ell.$$

Case 2. $r \ge b \vee \bar{b}$:

$$\widehat{u}[r,\infty) = u[r,\infty) \le v[r,\infty) + \ell = \widehat{v}[r,\infty) + \ell.$$

Case 3. $\bar{b} \le b$ and $a \le r < b$:

$$\widehat{u}[r,\infty) = u[b,\infty) \le v[b,\infty) + \ell = \widehat{v}[b,\infty) + \ell \le \widehat{v}[r,\infty) + \ell.$$

Case 4. $b < \bar{b}$ and $a \le r < \bar{b}$: Note that

$$\widehat{u}[b,\infty) = \|u\|_1 - (\Delta + \delta), \qquad \widehat{v}[\bar{b},\infty) = \|v\|_1 - (\Delta + \delta).$$

Moreover, $\|u\|_1 \le \|v\|_1 + \ell$. Hence, $\widehat{u}[b,\infty) \le \widehat{v}[\bar{b},\infty) + \ell$. Therefore,

$$\widehat{u}[r,\infty) \le \widehat{u}[b,\infty) \le \widehat{v}[\bar{b},\infty) + \ell \le \widehat{v}[r,\infty) + \ell.$$

(v) Note that

$$C_{\widehat{\Delta},\delta} u = C_\delta v + z, \qquad C_{\Delta,\delta} u = C_{\Delta - \widehat{\Delta},\delta} v + z,$$

where

$$v = C_{\widehat{\Delta}} u, \qquad z = \widehat{C}_{\widehat{\Delta}} u.$$

Therefore, it suffices to prove that we have $C_{\Delta - \widehat{\Delta},\delta} v \preccurlyeq C_\delta v$. To this end, note that we have $C_{\Delta - \widehat{\Delta},\delta} v = v - w \in L_1(\mathbb{R}, \mathbb{R}_+)$, and $\|w\|_1 = \delta$. Therefore, we can use part (iii) of the lemma, by which $v - w \preccurlyeq C_\delta v$. This completes the proof.

(vi) First, let us compare u to $C_\Delta v$. If $r < \Lambda_\Delta(v)$, then

$$u[r,\infty) \le \|u\|_1 = \|C_\Delta v\|_1 + \Delta = C_\Delta v[r,\infty) + \Delta \le C_\Delta v[r,\infty) + \Delta'.$$

If $r \ge \Lambda_\Delta(v)$, then

$$u[r,\infty) \le v[r,\infty) + \Delta' = C_\Delta v[r,\infty) + \Delta'.$$

This shows that $u \preccurlyeq C_\Delta v \bmod \Delta'$. By part (ii) of the lemma, $C_\delta u \preccurlyeq C_{\Delta+\delta} v \bmod \Delta'$. Finally, $C_{\Delta+\delta} v \le C_{\Delta,\delta} v$ pointwise, hence $C_{\Delta+\delta} v \preccurlyeq C_{\Delta,\delta} v$. This proves the claim. □

3.3. *On the barrier method.* We refer to [10] for an exposition of the use of barriers for proving uniqueness of solutions to FBP. Here, we briefly describe the main idea behind their use in earlier work, and how their structure differs from this line in the present paper.

Barriers in earlier work. For simplicity, consider the N-BBM setting of [12]. Here, $\alpha = 0$, $\kappa = 1$, $J_t = t$. It is proved in [12], Theorem 5, that for any classical solution (u, ℓ) to the FBP, u is sandwiched between lower and upper barriers in the sense that

$$(S_\delta C_{1-e^{-\delta}})^n u_0 \preccurlyeq u(\cdot, n\delta) \preccurlyeq (C_{e^\delta - 1} S_\delta)^n u_0 \tag{3.9}$$

for all $\delta > 0$ and n. Uniqueness is then deduced by showing that there can be at most one element that, for all δ and n, separates the lower from upper barriers. (A similar argument appeared, e.g., in [10], Theorem 3.14, for relaxed FBP solutions and in [4], Lemma 6.2, for solutions to the corresponding obstacle problem). The proof of (3.9) is based on a Feynman–Kac representation of u in terms of ℓ ([12], equation (6)), where the latter is assumed to be continuous (see, respectively, [10], Proposition 8.1, and [4], Proposition 3.1).

Next, if J is more general it is natural to expect, by analogy to (3.9), that

$$\prod_{i=0}^{n-1} (S_\delta C_{\hat{j}_i(\delta)}) u_0 \preccurlyeq u(\cdot, n\delta) \preccurlyeq \prod_{i=0}^{n-1} (C_{j_i(\delta)} S_\delta) u_0, \tag{3.10}$$

where $\hat{j}_i(\delta)$ and $j_i(\delta)$ depend on J, and reflect the fact that the mass removal rate varies. However, as mentioned in Remark 2.6, a Feynman–Kac representation for the density u of (2.10) is missing in the generality of Theorem 2.5 and, therefore, the argument given in the above references requires adaptation. We can recover the second inequality in (3.10) without this tool, but the first seems harder. We construct an alternative lower barrier for which we can prove the lower bound, by working with the operator $C_{\Delta,\delta}$ rather than C_δ. Here, our treatment considerably deviates from the above line.

Sketch of construction of barriers in this work. The construction of the barriers and the proof that they form bounds analogous to (3.9) and (3.10) appear in the next two subsections. Here, we sketch the main idea. Let (u, β) be a solution to (2.10). By equation (3.6), for $\delta > 0$, $u(\cdot, \delta) = v - h$, where

$$v = S_\delta \xi_0 + S * \alpha(\cdot, \delta), \qquad h = S * \beta(\cdot, \delta).$$

Let $j_0 = \|h\|_1$. Then the nonnegative function $u(\cdot, \delta)$ is obtained by removing from v the mass of size j_0 distributed according to h. If instead one removes from v the leftmost mass of size j_0, as shown in Figure 1(a), then the resulting function $C_{j_0} v$ satisfies $u = v - h \preccurlyeq C_{j_0} v$. Next, by Lemma 3.5, both S_δ and C_{j_0} preserve the order, and the argument can be iterated, providing an upper barrier at times $n\delta$ for all n. In the case $\alpha = 0$, this barrier takes the form of the right-hand side of (3.10).

Next, we sketch the lower barriers. Figure 1(b) shows in blue a mass of size j_0 located ε away from the leftmost mass of size δ. If ε is fixed while δ and j_0 are sufficiently small, then one can show that most of the mass of h (red) is to the left of the mass in blue. Removing from v the mass marked in blue thus gives a lower barrier for u, up to an error term, which can be made small. While this is a valid statement, it is not useful for us because the operator that cuts away the mass marked in blue does not preserve the order, and as a result the inequality cannot be iterated. However, one can use instead the operator C_{Δ, j_0} that leaves mass Δ on the left and then cuts away mass j_0, as shown in blue in Figure 1(c). With an a priori bound on the density, the statement regarding negligible mass in red reaching the mass in blue is still valid here. Since, by Lemma 3.5, this operator preserves the order, the argument can be iterated. As we will show, the resulting error term can be controlled.

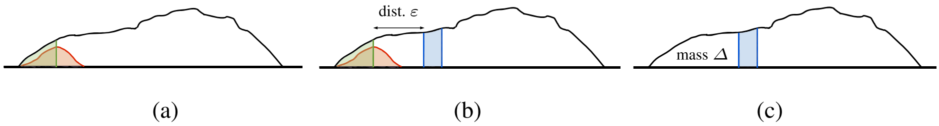

FIG. 1. (*a*) *The solution is given by* v (*black*) *minus* h (*red*); *upper barrier is obtained by removing the leftmost mass of size* $j_0 = \|h\|_1$ *from* v (*green*). (*b*) *Removing mass of size* j_0 (*blue*), *located* ε *away from the mass in green*. (*c*) *Removing instead mass of size* j_0 (*blue*) *after leaving mass of size* Δ *on the left, to obtain a lower barrier which can be iterated.*

3.4. *Upper barriers.* We now give the precise definitions. The upper barriers are defined using a removal mechanism that operates at times $n\delta$, $n \in \mathbb{N}$, for $\delta > 0$ fixed. Consider the time interval $[(n-1)\delta, n\delta]$. The mass injected during this interval adds the term $S * \alpha(\cdot, n\delta; (n-1)\delta)$ to the density at $n\delta$. Hence, let the "paste" operator be defined, for $u \in L_1(\mathbb{R}, \mathbb{R}_+)$, by

$$P_n^{(\delta)} u = u + S * \alpha(\cdot, n\delta; (n-1)\delta).$$

The mass removed during the said interval is of size $J_{n\delta} - J_{(n-1)\delta}$. However, more relevant is the size that this mass would grow to be had it not been removed, namely

$$j_n(\delta) := \int_{[(n-1)\delta, n\delta]} e^{\kappa(n\delta - s)}\, dJ_s.$$

Accordingly, let $C_n^{(\delta)} = C_{j_n(\delta)}$.

The upper barriers are defined for each $\delta > 0$ and $n \in \mathbb{N}$ by setting $u_0^{(\delta,+)} = \xi_0$ and

$$u_{n\delta}^{(\delta,+)} = C_n^{(\delta)} P_n^{(\delta)} S_\delta u_{(n-1)\delta}^{(\delta,+)}, \quad n \in \mathbb{N}.$$

Note that for $n = 1$ and $n \geq 2$, the function $S_\delta u_{(n-1)\delta}^{(\delta,+)}$ above is defined via (3.5) and respectively (3.4). For the barriers to be well-defined, one must have

$$P_n^{(\delta)} S_\delta u_{(n-1)\delta}^{(\delta,+)} \in H_{j_n(\delta)}. \tag{3.11}$$

We sometimes use the notation u_t for $u(\cdot, t)$ as we do in the following.

PROPOSITION 3.6. *Fix* $\delta > 0$. *Then* (3.11) *holds for all* $n \in \mathbb{N}$, *and consequently, the upper barriers are well-defined. Moreover, let* (u, β) *be a solution to* (2.10). *Then for* $n \in \mathbb{N}$,

$$u_{n\delta} \preccurlyeq u_{n\delta}^{(\delta,+)}.$$

PROOF. Recall that by Assumption 2.3, $m_t > 0$ for all t, where m_t is given by (2.6). The L_1 norm of $u^{(\delta,+)}$ satisfies

$$\|u_{n\delta}^{(\delta,+)}\|_1 = \|u_{(n-1)\delta}^{(\delta,+)}\|_1 e^{\kappa\delta} + \int_{[(n-1)\delta, n\delta]} e^{\kappa(n\delta - s)}\, dI_s - \int_{[(n-1)\delta, n\delta]} e^{\kappa(n\delta - s)}\, dJ_s,$$

so long as the right-hand side above is positive. By induction on n, this expression gives $\|u_{n\delta}^{(\delta,+)}\|_1 = m_{n\delta} > 0$, completing the proof of the first assertion. Note by a simple induction argument, the definition of $j_n(\delta)$ and Lemma 3.3, that $\|u_{n\delta}\|_1 = \|u_{n\delta}^{(\delta,+)}\|_1$.

The main claim will also be proved by induction. For $n - 1 \geq 1$, assume $f \preccurlyeq g$ where $f = u_{(n-1)\delta}$ and $g = u_{(n-1)\delta}^{(\delta,+)}$; for $n - 1 = 0$, $f = g = \xi_0$. Write C, P and S for $C_n^{(\delta)}$, $P_n^{(\delta)}$ and S_δ, respectively. We have $u_{n\delta}^{(\delta,+)} = CPSg$. Moreover, by Lemma 3.3, $u_{n\delta} = PSf - h$, where

$$h(y) = \int_{\mathbb{R} \times [(n-1)\delta, n\delta]} \mathfrak{s}_{n\delta - s}(x, y) \beta(dx, ds).$$

Hence, the proof will be complete once $PSf - h \preccurlyeq CPSg$ is shown.

By Lemma 3.5(i, ii), C preserves the order $\preccurlyeq$ and one has $S\tilde{u} \preccurlyeq S\tilde{v}$ whenever $\tilde{u} \preccurlyeq \tilde{v}$ and $\|\tilde{u}\|_1 = \|\tilde{v}\|_1$. It is trivial that this is also true for P. Denote $w = PSf$. Suppose one shows $w - h \preccurlyeq Cw$. Then

$$w - h \preccurlyeq Cw = CPSf \preccurlyeq CPSg,$$

where in the last inequality one uses $Sf = Sg$ for $n-1=0$ and $\|f\|_1 = \|g\|_1$ for $n-1 \geq 1$. Thus, the proof would be complete.

It thus suffices to show $w - h \preccurlyeq Cw$. The function $w - h$ is nonnegative (as required by the definition of a solution) and $\int h = j_n(\delta)$. Hence, $w - h \preccurlyeq Cw$ by Lemma 3.5(iii), and the proof is complete. □

3.5. *Lower barriers*. The lower barriers are defined for $\hat{\delta} = (\Delta, \delta) \in (0,\infty)^2$ and $n \in \mathbb{N}$ as follows. Let

$$C_n^{(\hat{\delta})} = C_{\Delta, j_n(\delta)}.$$

Set $u_0^{(\hat{\delta},-)} = \xi_0$ and $\ell_{0,\hat{\delta}} = 0$, and for $n \in \mathbb{N}$,

$$\begin{aligned} u_{n\delta}^{(\hat{\delta},-)} &= C_n^{(\hat{\delta})} P_n^{(\delta)} S_\delta u_{(n-1)\delta}^{(\hat{\delta},-)}, \\ \ell_{n,\hat{\delta}} &= e^{\kappa\delta}\ell_{n-1,\hat{\delta}} + \begin{cases} j_n(\delta) & \text{if } (n-1)\delta < t_0, \\ e^{-\Delta^5/\delta} j_n(\delta) & \text{if } (n-1)\delta \geq t_0, \end{cases} \end{aligned} \tag{3.12}$$

where $t_0 > 0$ is fixed. Once again, for the definition to be valid, one must assure that for all $n \in \mathbb{N}$,

$$P_n^{(\delta)} S_\delta u_{(n-1)\delta}^{(\hat{\delta},-)} \in H_{\Delta + j_n(\delta)}. \tag{3.13}$$

PROPOSITION 3.7. *For $\Delta \in (0, \varepsilon_0)$ and $\delta > 0$, (3.13) holds for all n, and consequently, the lower barriers are well-defined. Moreover, let $0 < t_0 < T$ be given. Then there exists $\Delta_0 \in (0, \varepsilon_0)$ such that for every $\Delta \in (0, \Delta_0)$ there exists $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$ and $n \in \mathbb{N}$ satisfying $n\delta \leq T$ one has*

$$u_{n\delta}^{(\hat{\delta},-)} \preccurlyeq u_{n\delta} \bmod \ell_{n,\hat{\delta}}$$

whenever (u, β) is a solution to (2.10). Furthermore, for $n \in \mathbb{N}$, $n\delta \leq T$ one has

$$\ell_{n,\hat{\delta}} \leq e^{\kappa(T+\delta)}\big(J_{t_0+\delta} + e^{-\Delta^5/\delta} J_T\big). \tag{3.14}$$

Given $\gamma \in \mathcal{M}_{+,\mathrm{loc}}(\mathbb{R} \times \mathbb{R}_+)$ and $[t_1, t_2] \subset \mathbb{R}_+$, the supremum of the support of the measure $\tilde{\gamma}(\cdot) = \gamma(\cdot \times [t_1, t_2])$ is denoted by $\rho^*(\gamma; [t_1, t_2])$. Recall c_* from (3.2) and denote $c^* = 2/c_*$.

LEMMA 3.8. *Given a solution (u, β) to (2.10), $\delta > 0$ and $n \in \mathbb{N}$, let*

$$\rho_{n,\delta} = \rho^*\big(\beta; [(n-1)\delta, n\delta]\big).$$

Then for $n \geq 2$,

$$\rho_{n,\delta} \leq b(n, \delta, u_{(n-1)\delta}) := \Lambda_{c^* j_n(\delta)}(u_{(n-1)\delta}), \quad \textit{provided } u_{(n-1)\delta} \in H_{c^* j_n(\delta)} \textit{ and } j_n(\delta) > 0.$$

PROOF. We consider only $n = 2$; the proof for $n > 2$ is similar. Fix δ and a solution (u, β). Arguing by contradiction, assume $\rho_{2,\delta} > b = b(2, \delta, u_\delta) = \Lambda_{c^* j_2(\delta)}(u_\delta)$ (the latter is well-defined and finite by the assumption $u_\delta \in H_{c^* j_2(\delta)}$). Hence, $\beta((b,\infty) \times [\delta, 2\delta]) > 0$. Because β does not charge $\mathbb{R} \times \{\delta\}$, it follows that $\theta := \beta((b,\infty) \times (\delta, 2\delta]) > 0$.

Let us show using (2.10)(ii) that there exists $t \in (\delta, 2\delta]$ such that $U(b,t) = 0$. If this statement is false, namely $U(b,t) > 0$ for all $t \in (\delta, 2\delta]$, then

$$\beta(U > 0) \geq \int_{(b,\infty)\times(\delta,2\delta]} 1_{\{U(x,t)>0\}}\beta(dx,dt) \geq \int_{(b,\infty)\times(\delta,2\delta]} 1_{\{U(b,t)>0\}}\beta(dx,dt) = \theta > 0,$$

contradicting (2.10)(ii).

Fix such $t \in (\delta, 2\delta]$. We now appeal to identity (3.7) with $\tau = \delta$. Denoting the last term there by $h(y) = \int_{\mathbb{R}\times[\delta,t]} \mathfrak{s}_{t-s}(x,y)\beta(dx,ds)$,

$$\int_{-\infty}^{b} h(y)\,dy \leq \int_{-\infty}^{\infty} h(y)\,dy = \int_{[\delta,t]} e^{\kappa(t-s)}\,dJ_s \leq \int_{[\delta,2\delta]} e^{\kappa(2\delta-s)}\,dJ_s = j_2(\delta).$$

Since $U(b,t) = 0$, we have for the first term in (3.7),

$$\int_{-\infty}^{b}\int_{\mathbb{R}} \mathfrak{s}_{t-\delta}(x,y)u(x,\delta)\,dx\,dy \leq \int_{-\infty}^{b} h(y)\,dy \leq j_2(\delta).$$

Using (3.2), $\int_{-\infty}^{x} \mathfrak{s}_{t-\delta}(x,y)\,dy \geq \int_{-\infty}^{x} \mathfrak{p}_{t-\delta}(x,y)\,dy \geq c_*$ for all x, hence

$$\begin{aligned} j_2(\delta) &\geq \int_{x\in(-\infty,b]}\int_{y\in(-\infty,b]} \mathfrak{s}_{t-\delta}(x,y)u(x,\delta)\,dy\,dx \\ &\geq c_*\int_{-\infty}^{b} u(x,\delta)\,dx = c_*c^* j_2(\delta) = 2j_2(\delta), \end{aligned}$$

a contradiction due to the assumption $j_2(\delta) > 0$. This proves the claim. $\square$

PROOF OF PROPOSITION 3.7. The proof of (3.13) is similar to the proof of the analogous statement from Proposition 3.6. Also as in that proof, the L_1 norm of the solution and the barrier at $n\delta$ are equal. As for (3.14), fix $0 < t_0 < T$. Denoting $\chi_n = 1_{(n-1)\delta<t_0} + e^{-\Delta^5/\delta}$, it follows from (3.12) that

$$\ell_{n,\hat{\delta}} \leq e^{\kappa\delta}\ell_{n-1,\hat{\delta}} + j_n(\delta)\chi_n.$$

By induction, $\ell_{n,\hat{\delta}} \leq e^{n\kappa\delta}\sum_{i=1}^{n} j_i(\delta)\chi_i$. Using $j_n(\delta) \leq e^{\kappa\delta}(J_{n\delta} - J_{(n-1)\delta})$ and $n\delta \leq T$,

$$\ell_{n,\hat{\delta}} \leq e^{\kappa(T+\delta)}\big(J_{t_0+\delta} + e^{-\Delta^5/\delta}J_T\big),$$

as claimed.

We turn to the main assertion. Assume that $\Delta < \varepsilon_0/2$. Arguing by induction, assume that $f \preccurlyeq g \bmod \ell_{n-1,\hat{\delta}}$, where

$$f = u_{(n-1)\delta}^{(\hat{\delta},-)}, \qquad g = u_{(n-1)\delta},$$

when $n-1 \geq 1$ and $f = g = \xi_0$ when $n-1 = 0$. Write C, P and S for $C_n^{(\hat{\delta})}$, $P_n^{(\delta)}$ and S_δ, respectively. Then $u_{n\delta}^{(\hat{\delta},-)} = CPSf$, and by Lemma 3.3, $u_{n\delta} = PSg - h$, where

$$h(y) = \int_{\mathbb{R}\times[(n-1)\delta,n\delta]} \mathfrak{s}_{n\delta-s}(x,y)\beta(dx,ds).$$

By Lemma 3.5(i), $PSf \preccurlyeq PSg \bmod e^{\kappa\delta}\ell_{n-1,\hat{\delta}}$. If $(n-1)\delta < t_0$, we therefore have, for any r,

$$\begin{aligned} u_{n\delta}^{(\hat{\delta},-)}[r,\infty) &= CPSf[r,\infty) \leq PSf[r,\infty) \\ &\leq PSg[r,\infty) + e^{\kappa\delta}\ell_{n-1,\hat{\delta}} \end{aligned}$$

$$\leq PSg[r,\infty) - h[r,\infty) + \|h\|_1 + e^{\kappa\delta}\ell_{n-1,\hat{\delta}}$$

$$= u_{n\delta}[r,\infty) + j_n(\delta) + e^{\kappa\delta}\ell_{n-1,\hat{\delta}}$$

$$= u_{n\delta}[r,\infty) + \ell_{n,\hat{\delta}},$$

which gives the claimed estimate.

In what follows, $(n-1)\delta \geq t_0$. In particular, $n \geq 2$. In view of the lower bound on $m_t = \|u(\cdot,t)\|_1$, $t \in [0,T]$ and the continuity of J, we may assume that δ is so small that the condition $u_{(n-1)\delta} \in H_{c^* j_n(\delta)}$ holds for all $n \geq 2$, $n\delta \leq T$. As a result, the bound asserted in Lemma 3.8 is valid provided merely that $j_n(\delta) > 0$. Moreover, by Lemma 3.3 there exists a constant c_∞ such that for any solution (u,β), $\|u(\cdot,t)\|_\infty < c_\infty$, $t \in [t_0,T]$. Using the induction assumption and Lemma 3.5(iv),

$$u_{n\delta}^{(\hat{\delta},-)} = CPSf \preccurlyeq CPSg \bmod e^{\kappa\delta}\ell_{n-1,\hat{\delta}}.$$

Denote $w = PSg$. Suppose

$$Cw \preccurlyeq w - h \bmod \varepsilon \quad \text{where } \varepsilon = \ell_{n,\hat{\delta}} - e^{\kappa\delta}\ell_{n-1,\hat{\delta}} = e^{-\Delta^5/\delta} j_n(\delta). \tag{3.15}$$

Then $u_{n\delta}^{(\hat{\delta},-)} \preccurlyeq w - h = u_{n\delta} \bmod \ell_{n,\hat{\delta}}$, which completes the proof. It remains to show (3.15).

First, if $j_n(\delta) = 0$, then $\varepsilon = 0$ and (3.15) holds because $C = \mathtt{id}$, $h = 0$ and therefore $Cw = w = w - h$. Next, assume $j_n(\delta) > 0$. Denote $j = j_n(\delta)$ and $b = \Lambda_{c^* j_n(\delta)}(g) = \Lambda_{c^* j}(g)$. By Lemma 3.8, $\rho_{n,\delta} \leq b$. Write $h = h_1 + h_2$, where

$$h_1(y) = h(y)1_{\{y > b+\Delta^2\}}, \qquad h_2(y) = h(y)1_{\{y \leq b+\Delta^2\}}.$$

Because $\rho_{n,\delta} \leq b$, we have

$$h(y) = \int_{(-\infty,b]\times[(n-1)\delta,n\delta]} \mathfrak{s}_{n\delta-s}(x,y)\beta(dx,ds).$$

Without loss of generality, $e^{\kappa\delta} < 2$, thus $\mathfrak{s}_t \leq 2\mathfrak{p}_t$ for $t \leq \delta$. Hence, in view of Lemma 3.2, $\mathfrak{s}_t(0,[a,\infty)) \leq c_3 e^{-c_4 a^2/t}$ for $a > t^{1/2} > 0$, where $c_3, c_4 > 0$ depend only on $\hat{c}_1$, $\hat{c}_2$ of the lemma. This gives

$$\|h_1\|_1 \leq c_3 \int_{(-\infty,b]\times[(n-1)\delta,n\delta]} e^{-c_4\Delta^4/(n\delta-s)}\beta(dx,ds) \leq c_3 e^{-c_4\Delta^4/\delta} j,$$

provided $\delta < \Delta^4$. If we further require $\Delta < c_4$, then for all sufficiently small δ,

$$\|h_1\|_1 \leq e^{-\Delta^5/\delta} j = \varepsilon. \tag{3.16}$$

Recall that $\|h\|_1 = j$ and let $q \in (0,1]$ be defined by $\|h_2\|_1 = qj$. Then $q \geq 1 - e^{-\Delta^5\delta}$. Let us argue that it suffices to show

$$b + \Delta^2 = \Lambda_{c^* j}(g) + \Delta^2 \leq \Lambda_\Delta(w) \tag{3.17}$$

in order to prove (3.15). By definition, h_2 is supported to the left of $b + \Delta^2$. On the other hand, $C_{\Delta,qj} w = w - \tilde{h}$, where $\|\tilde{h}\|_1 = qj$ and $\tilde{h}$ is supported to the right of $\Lambda_\Delta(w)$. Thus, using $\|\tilde{h}\|_1 = \|h_2\|_1$, it follows from (3.17) that $C_{\Delta,qj} w \preccurlyeq w - h_2 = w - h + h_1$. In view of (3.16), this gives

$$C_{\Delta,qj} w \preccurlyeq w - h \bmod \varepsilon.$$

Because $Cw = C_{\Delta,j} w \leq C_{\Delta,qj} w$ pointwise, one has $Cw \preccurlyeq C_{\Delta,qj} w$. Hence, (3.15) follows.

It remains to show (3.17). Because $t_0 \le (n-1)\delta \le T$, the bound $\|g\|_\infty \le c_\infty$ is valid. By making Δ smaller if needed, assume $2c_\infty\Delta^2 < \Delta/6$. Then for all δ so small that $c^* j < \Delta/6$ (simultaneously over n),

$$\Lambda_{\Delta/3}(g) \ge \Lambda_{c^* j}(g) + 2\Delta^2. \tag{3.18}$$

Next, we argue that for all small δ,

$$\Lambda_{2\Delta/3}(Sg) \ge \theta := \Lambda_{\Delta/3}(g) - \Delta^2. \tag{3.19}$$

To show this, we must to show $(Sg)(-\infty,\theta] \le 2\Delta/3$. Let $g = g_1 + g_2 = \widehat{C}_{\Delta/3}g + C_{\Delta/3}g$. Because $\|g_1\|_1 = \Delta/3$, we have $\|Sg_1\|_1 \le e^{\kappa\delta}\Delta/3 \le \Delta/2$, and

$$(Sg)(-\infty,\theta] = (Sg_1)(-\infty,\theta] + (Sg_2)(-\infty,\theta] \le \Delta/2 + \|g_2\|_1 e^{\kappa\delta}\mathfrak{p}_\delta\big(x_0, (-\infty,\theta]\big),$$

where $x_0 = \Lambda_{\Delta/3}(g)$, owing to the fact that $\mathfrak{p}_\delta(y, (-\infty,\theta])$ is monotone decreasing in y for $y > \theta$. Recalling that $\|g\|_1 = \|u_{(n-1)\delta}\|_1 = m_{(n-1)\delta} \le \|m\|_T^*$ and using again Lemma 3.2, the last term in the above display is bounded by

$$2\hat{c}_1\|m\|_T^* \int_{-\infty}^{x_0-\Delta^2} \delta^{-1/2} e^{-\hat{c}_2(x_0-y)^2\delta^{-1}}\,dy,$$

which is smaller than $\Delta/6$ for all sufficiently small δ. This shows $(Sg)(-\infty,\theta] \le 2\Delta/3$, hence (3.19).

For $\pi := S * \alpha(\cdot, n\delta; (n-1)\delta)$, we have $\|\pi\|_1 \le e^{\kappa\delta}(I_{n\delta} - I_{(n-1)\delta})$. Hence, for all small δ, $\|\pi\|_1 < \Delta/3$. As a result,

$$\Lambda_\Delta(w) = \Lambda_\Delta(PSg) = \Lambda_\Delta(Sg + \pi) \ge \Lambda_{2\Delta/3}(Sg).$$

Combining this with (3.18) and (3.19) gives (3.17), and the proof is complete. □

3.6. *Proof of uniqueness*. The last step is showing that the lower and upper barriers become close upon taking $\delta \to 0$ then $\Delta \to 0$, and finally $t_0 \to 0$.

PROPOSITION 3.9. *Fix $0 < t_0 < T$. Let Δ_0 and $\delta_0 = \delta_0(\Delta_0)$ be as in Proposition 3.7. Then for $\Delta \in (0, \Delta_0)$, $\delta \in (0, \delta_0)$ and $n \in \mathbb{N}$, $n\delta \le T$, one has*

$$u_{n\delta}^{(\delta,+)} \preccurlyeq u_{n\delta}^{(\hat{\delta},-)} \operatorname{mod} e^{n\kappa\delta}\Delta.$$

PROOF. We argue by induction. First, recall $u_0^{(\pm)} = \xi_0$. Next, assume that we have $u_{(n-1)\delta}^{(\delta,+)} \preccurlyeq u_{(n-1)\delta}^{(\hat{\delta},-)} \operatorname{mod} e^{(n-1)\kappa\delta}\Delta$. Then

$$P_n^{(\delta)} S_\delta u_{(n-1)\delta}^{(\delta,+)} \preccurlyeq P_n^{(\delta)} S_\delta u_{(n-1)\delta}^{(\hat{\delta},-)} \operatorname{mod} e^{n\kappa\delta}\Delta,$$

where, for $n-1=0$, this is true because both sides of the inequality are equal, and otherwise this is a consequence of the induction assumption and Lemma 3.5(i), recalling that the L_1 norm of the upper and lower barriers are equal for each n. For the same reason, Lemma 3.5(vi) also applies, and gives

$$C_n^{(\delta)} P_n^{(\delta)} S_\delta u_{(n-1)\delta}^{(\delta,+)} \preccurlyeq C_n^{(\hat{\delta})} P_n^{(\delta)} S_\delta u_{(n-1)\delta}^{(\hat{\delta},-)} \operatorname{mod} e^{n\kappa\delta}\Delta,$$

that is, $u_{n\delta}^{(\delta,+)} \preccurlyeq u_{n\delta}^{(\hat{\delta},-)} \operatorname{mod} e^{n\kappa\delta}\Delta$. This completes the proof. □

PROOF OF THEOREM 3.1. Once uniqueness is established for the u component of the solution (u,β), uniqueness of the β component follows from (2.11). By Remark 3.4, it suffices to prove uniqueness of solutions (u,β) in which u is the version given by Lemma 3.3.

To show uniqueness of the u component, argue by contradiction and assume that (u^i,β^i), $i=1,2$ are two solutions where u^1 and u^2 are distinct. Then there exist $t>0$ and $r\in\mathbb{R}$ such that, say, $u^1_t[r,\infty)<u^2_t[r,\infty)$. Fix such t and r. Denote $\delta_n=tn^{-1}$ for $n\in\mathbb{N}$. Let $0<t_0<t$. Then, by Propositions 3.6 and 3.7, for every small $\Delta>0$ there exists n_0 such that for $n>n_0$,

$$u_t^{(\Delta,\delta_n,-)}[r,\infty)-\ell_{n,\Delta,\delta_n}\le u_t^1[r,\infty)<u_t^2[r,\infty)\le u_t^{(\Delta,+)}[r,\infty).$$

By Proposition 3.9,

$$u_t^{(\Delta,+)}[r,\infty)\le u_t^{(\Delta,\delta_n,-)}[r,\infty)+e^{\kappa t}\Delta.$$

Using these two inequalities and then the bound from (3.14),

$$0<u_t^2[r,\infty)-u_t^1[r,\infty)\le e^{\kappa t}\Delta+\ell_{n,\Delta,\delta_n}\le e^{\kappa t}\Delta+e^{\kappa(t+\delta_n)}\big(J_{t_0+\delta_n}+e^{-\Delta^5/\delta_n}J_t\big).$$

On taking $n\to\infty$, then $\Delta\downarrow 0$ and finally $t_0\downarrow 0$, the expression on the right converges to zero, a contradiction. □

4. Convergence. In this section, the convergence result is proved based on the FBP uniqueness, yielding the proof of Theorem 2.5. Throughout, the assumptions of Theorem 2.5 hold. The main steps are as follows. In Lemma 4.1, the normalized processes are shown to satisfy a version of equation (2.9) with an error term. Lemma 4.2 establishes tightness of these processes. Existence of a measurable density for any limit point of the sequence $\bar{\xi}^N$ is shown in Lemma 4.3. In Lemma 4.4, the final, crucial step shows that the complementarity condition is preserved under the limit.

4.1. *Limit laws and the parabolic equation.* This subsection contains the first three of the aforementioned steps toward convergence. Some additional notation used here is as follows. Let $J_t^{\#,N}=\beta^N(\mathbb{R}\times[0,t])$ denote the counting process for removals, and note that, by construction, $J_t^{\#,N}=J_t^N$ holds on the event $\{\inf_{s\le t}m_s^N\ge 1\}$ (recall the remark after (2.4)). Let

$$Y_t^N=\sum_{i\in\mathcal{S}}1_{\{\sigma^i\le t\}}\big(\pi^i_{t\wedge\tau^i}-\pi^i_{\sigma^i}\big)$$

be the number of births during time $[0,t]$, and let its macroscopic counterpart be given by $Y_t=\kappa\int_0^t m_s\,ds$. Denote $R_T^N=N+I_T^N$ and note that the set $\mathcal{R}_T^N=\{(j,0):j\le R_T^N\}$ consists of all root particles appearing by time T.

Recall that a solution to (2.10)(i, iv) is defined via (2.11). The relation of the particle system to this equation is established by showing that if (ξ,β) is a limit point of $(\bar{\xi}^N,\bar{\beta}^N)$ then

$$-\int_0^\infty\langle\partial_t\varphi+\mathcal{L}\varphi,\xi_t\rangle\,dt=\int_{\mathbb{R}}\varphi(\cdot,0)\,d\xi_0+\int_{\mathbb{R}\times\mathbb{R}_+}\varphi\,d\alpha-\int_{\mathbb{R}\times\mathbb{R}_+}\varphi\,d\beta. \tag{4.1}$$

LEMMA 4.1. (i) *Let $\varphi\in C_c^\infty(\mathbb{R}\times\mathbb{R}_+)$ and let T be such that $\varphi(\cdot,t)=0$ for all $t\ge T$. Then*

$$-\int_0^\infty\langle(\partial_t\varphi+\mathcal{L}\varphi)(\cdot,t),\bar{\xi}_t^N\rangle\,dt=\langle\varphi(\cdot,0),\bar{\xi}_0^N\rangle+\int_{\mathbb{R}\times\mathbb{R}_+}\varphi\,d\bar{\alpha}^N-\int_{\mathbb{R}\times\mathbb{R}_+}\varphi\,d\bar{\beta}^N+\bar{M}_T^N,$$

where $\bar{M}_t^N$ is an $\{\mathcal{F}_t^N\}$-martingale starting at zero, with quadratic variation

$$[\bar{M}^N]_t\le cN^{-1}\bigg(\int_0^t\bar{m}_s^N\,ds+\bar{Y}_t^N\bigg), \tag{4.2}$$

where c depends only on φ and $\mathfrak{c}$.

(ii) *One has*

$$\bar{m}_t^N = 1 + \kappa \int_0^t \bar{m}_s^N \, ds + \bar{I}_s^N - \bar{J}_t^{\#,N} + \bar{M}_t^{\#,N},$$

where, with c as above, $\bar{M}_t^{\#,N}$ *is an* $\{\mathcal{F}_t^N\}$*-martingale starting at zero, and*

$$[\bar{M}^{\#,N}]_t \le c N^{-1} \bar{Y}_t^N.$$

(iii) *As* $N \to \infty$, $(\bar{m}^N, \bar{Y}^N, \bar{I}^N, \bar{J}^N, \bar{J}^{\#,N}) \to (m, Y, I, J, J)$ *in probability.*

(iv) *Suppose that* $(\xi_0, \xi, \alpha, \beta)$ *is a subsequential limit of* $(\bar{\xi}_0^N, \bar{\xi}^N, \bar{\alpha}^N, \bar{\beta}^N)$. *Then the former tuple satisfies* (4.1) *a.s.*

PROOF. (i) Note that σ^i and τ^i are $\{\mathcal{F}_t^N\}$-stopping times and recall that B^i and $\hat{\pi}^i$ are martingales on this filtration. By Itô's lemma, for each $i \in \mathcal{S}$, on the event $\{t \ge \sigma^i\}$,

$$\begin{aligned}
\varphi(X^i_{t\wedge\tau^i}, t \wedge \tau^i) &= \varphi(x^i, \sigma^i) + \int_{\sigma^i}^{t\wedge\tau^i} (\partial_t \varphi + \mathfrak{b}\partial_x \varphi + \mathfrak{a}\partial_x^2 \varphi)(X_s^i, s)\, ds \\
&\quad + \int_{\sigma^i}^{t\wedge\tau^i} (\mathfrak{c}\partial_x \varphi)(X_s^i, s)\, dB_s^i \qquad (4.3)\\
&= \varphi(x^i, \sigma^i) + \int_{\sigma^i}^{t\wedge\tau^i} (\partial_t \varphi + \mathfrak{b}\partial_x \varphi + \mathfrak{a}\partial_x^2 \varphi)(X_s^i, s)\, ds + M_t^{N,i,1},
\end{aligned}$$

where

$$M_t^{N,i,1} = 1_{\{\sigma^i \ge t\}} \int_{\sigma^i}^{t\wedge\tau^i} (\mathfrak{c}\partial_x \varphi)(X_s^i, s)\, dB_s^i.$$

Given i, the sum of evaluations of φ over birth location-epochs of particles born directly from particle i between time 0 and t is given by

$$1_{\{t \ge \sigma^i\}} \int_{\sigma^i}^{t\wedge\tau^i} \varphi(X_s^i, s)\, d\pi_s^i.$$

Summing this expression over $i \in \mathcal{S}$ gives the sum of evaluations of φ over all birth location-epochs during that time interval, that is,

$$\begin{aligned}
\sum_{i=(j,k)\in\mathcal{S}_t^N,\, k\ge 1} \varphi(x^i, \sigma^i) &= \sum_{i\in\mathcal{S}} 1_{\{t\ge\sigma^i\}} \int_{\sigma^i}^{t\wedge\tau^i} \varphi(X_s^i, s)\, d\pi_s^i \\
&= \sum_{i\in\mathcal{S}} 1_{\{t\ge\sigma_i\}} \left[\int_{\sigma^i}^{t\wedge\tau^i} \varphi(X_s^i, s)\kappa\, ds + M_t^{N,i,2} \right],
\end{aligned}$$

where the index set on the left corresponds to births by time t, and

$$M_t^{N,i,2} = 1_{\{t\ge\sigma^i\}} \int_{\sigma^i}^{t\wedge\tau^i} \varphi(X_s^i, s)\, d\hat{\pi}_s^i.$$

Therefore, summing (4.3) over all i such that $\sigma^i \le t$ and normalizing gives

$$\begin{aligned}
\int_{\mathbb{R}\times[0,t]} \varphi\, d\bar{\beta}^N &= \int_{\mathbb{R}} \varphi(\cdot, 0)\, d\bar{\xi}_0^N + \int_{\mathbb{R}\times[0,t]} \varphi\, d\bar{\alpha}^N \\
&\quad + \int_0^t \int_{\mathbb{R}} (\partial_t \varphi + \mathcal{L}\varphi)(x, t) \bar{\xi}_t^N(dx)\, dt + \bar{M}_t^N,
\end{aligned}$$

where

$$\bar{M}_t^N = N^{-1} M_t^N, \qquad M_t^N = \sum_{i \in \mathcal{S}} (M_t^{N,i,1} + M_t^{N,i,2}).$$

Take $t = T$ and replace the integration range $\mathbb{R} \times [0, T]$ to $\mathbb{R} \times \mathbb{R}_+$ recalling that $\varphi(x, t) = 0$ for $t > T$. The bound (4.2) follows from $[M^{N,i,1}]_t \le 1_{\{t \ge \sigma_i\}} \|\mathfrak{c}\|_\infty^2 \|\partial_x \varphi\|_\infty^2 (t \wedge \tau^i - \sigma^i)$ and $[M^{N,i,2}]_t \le 1_{\{t \ge \sigma_i\}} \|\varphi\|_\infty^2 (\pi^i_{t \wedge \tau^i} - \pi^i_{\sigma^i})$ and the identities

$$\sum_{i \in \mathcal{S}} 1_{\{t \ge \sigma^i\}} (t \wedge \tau^i - \sigma^i) = \int_0^t \xi_s^N(\mathbb{R})\, ds = \int_0^t m_s^N\, ds$$

and

$$\sum_{i \in \mathcal{S}} 1_{\{t \ge \sigma^i\}} (\pi^i_{t \wedge \tau^i} - \pi^i_{\sigma^i}) = Y_t^N. \tag{4.4}$$

(ii) We have $m_t^N = N + I_t^N + Y_t^N - J_t^{\#,N}$ by the definition of these processes. By (4.4),

$$Y_t^N = \kappa \int_0^t m_s^N\, ds + \kappa M_t^{\#,N} \quad \text{where } M_t^{\#,N} = \sum_{i \in \mathcal{S}} 1_{\{t \ge \sigma^i\}} \int_{\sigma^i}^{t \wedge \tau^i} d\hat{\pi}_s^i. \tag{4.5}$$

The quadratic variation bound follows as in (i).

(iii) Fix T. Recall ε_0 from Assumption 2.3. Consider the $\{\mathcal{F}_t^N\}$-stopping time

$$\theta^N = \inf\{t : \bar{I}_t^N \ge I_T + 1 \text{ or } \bar{m}_t^N \le \varepsilon_0/2\}.$$

By (ii) and Gronwall's lemma, $\mathbb{E}[\bar{m}^N_{t \wedge \theta^N}] \le c$, where $c = c(T)$. Hence, by (4.5), one has $\mathbb{E}[\bar{Y}^N_{t \wedge \theta^N}] \le c$. Going back to (ii) gives $\mathbb{E}[[\bar{M}^{\#,N}]_{T \wedge \theta^N}] \to 0$, hence $\|\bar{M}^{\#,N}\|^*_{T \wedge \theta^N} \to 0$ in probability. For $t \le \theta^N$, one has $J_t^{\#,N} = J_t^N$. Thus, using (2.6) and again Gronwall's lemma, one has that $\|\bar{m}^N - m\|^*_{T \wedge \theta^N} \to 0$ in probability. By the definition of ε_0, this shows that $\mathbb{P}(\theta^N < T) \to 0$. Because T is arbitrary, this proves that $\bar{m}^N \to m$ and $\bar{J}^{\#,N} \to J$ in probability. As a result, by (4.5), $\bar{Y}^N \to Y$ in probability.

(iv) In view of (i), it suffices to show that $\|\bar{M}^N\|_T^* \to 0$ in probability. However, this is an immediate consequence of (4.2) and (iii). □

The following point is used in the proof of the next two lemmas. One can construct an additional particle system in which there are no removals, based on the same stochastic primitives as in the original system, except $\tilde{\eta}^l = \infty$ for all l in place of the original removal attempt times η^l. In this particle system, we use tilde notation for all the model ingredients, as in $\tilde{x}^i$, $\tilde{X}^i$, $\tilde{\sigma}^i$, $\tilde{\mathcal{S}}_t^N$, with one exception: instead of $\tilde{\xi}^N$, we write ζ^N (and $\bar{\zeta}^N$ for its normalized version). Thus, $\tilde{J}^N = 0$, $\tilde{\beta}^N = 0$ and $\tilde{\tau}^i = \infty$ for all i. As a consequence, the tilde system dominates the original system: For every $t \ge 0$, $\bar{\xi}_t^N \sqsubset \bar{\zeta}_t^N$.

LEMMA 4.2. *The sequence of laws of* $(\bar{\xi}^N, \bar{\beta}^N)$, $N \in \mathbb{N}$ *is tight. For every subsequential limit* (ξ, β), *one has* $\mathbb{P}(\xi \in C(\mathbb{R}_+, \mathcal{M}_+(\mathbb{R}))) = 1$.

PROOF. Both the J_1 topology over $D(\mathbb{R}_+, \mathbb{R})$ and the topology of local weak convergence we gave the space $\mathcal{M}_{+,\mathrm{loc}}(\mathbb{R} \times \mathbb{R}_+)$ are defined by convergence over finite time intervals. Hence, in this proof, we fix T and consider all processes (resp., measures) defined on $\mathbb{R}_+$ (on subsets of $\mathbb{R} \times \mathbb{R}_+$) as if they are defined on $[0, T]$ (on subsets of $\mathbb{R} \times [0, T]$), and with a slight abuse of notation still use the same notation. For example, β^N will denote the restriction of the original random measure β^N to subsets of $\mathbb{R} \times [0, T]$.

Moreover, recalling that $\bar{\alpha}^N(\mathbb{R}\times[0,T]) = \bar{I}^N_T \to I_T$ in probability, we may and will assume w.l.o.g. that the injections are truncated when their number reaches $c_I N$, where $c_I = I_T + 1$. Hence, for all N and $t\in[0,T]$, $I^N_t \le c_I N$ a.s. Similarly, the removal measure is assumed w.l.o.g. to be truncated when J^N reaches $c_J N$, $c_J = J_T + 1$.

Tightness may be argued separately for each component. Starting with $\bar{\beta}^N$, for $r>0$, denote $\mathbb{B}_r = [-r,r]\times[0,T]$, $\mathbb{B}^c_r = (\mathbb{R}\setminus[-r,r])\times[0,T]$. Since we have $|\bar{\beta}^N|\le c_J$, it suffices to show that for every $n\in\mathbb{N}$ there is r such that

$$\liminf_N \mathbb{P}\big(\bar{\beta}^N(\mathbb{B}^c_r) < n^{-1}\big) \ge 1-n^{-1}. \tag{4.6}$$

To this end, note that the removal space-time location of a particle is a point on the graph of the potential trajectory of that particle, hence

$$\beta^N(\mathbb{B}^c_r) \le U^N_r := \sum_{i\in\mathcal{S}^N_T} 1_{\{\|X^i\|^*_{[\sigma^i,T]}>r\}}. \tag{4.7}$$

If we let

$$\tilde{U}^N_r = \sum_{i\in\tilde{\mathcal{S}}^N_T} 1_{\{\|\tilde{X}^i\|^*_{[\tilde{\sigma}^i,T]}>r\}},$$

then for each N and r, the random variable U^N_r is dominated by $\tilde{U}^N_r$. This shows

$$\mathbb{P}\big(\bar{\beta}^N(\mathbb{B}^c_r)\ge n^{-1}\big) \le \mathbb{P}\big(\tilde{U}^N_r \ge Nn^{-1}\big). \tag{4.8}$$

Now, in the tilde system, the collection of family members descending from a root particle $i=(j,0)$ up to time T is denoted by $\tilde{\mathcal{S}}^{N,i}_T$, in accordance with (2.3). Let

$$\begin{aligned}\tilde{\mathcal{F}}^N_* &= \sigma\{I^N_T,\ (\tilde{x}^i,\tilde{\sigma}^i): i\in\mathcal{R}^N_T\}\\ &= \sigma\{I^N_T,\ (x^i,\sigma^i): i\in\mathcal{R}^N_T\},\end{aligned}$$

where the equality follows by construction. For $i\in\mathcal{R}^N_T$ and $\hat{i}\in\tilde{\mathcal{S}}^{N,i}_T$, let $\tilde{X}^{i,\hat{i}}_t$, $t\in[\tilde{\sigma}^i,T]$ denote the trajectory formed by $\tilde{X}^{\hat{i}}_t$ during its lifetime, and by the trajectories of its ancestors prior to its birth time (here as well $\tilde{\sigma}^i$ can be replaced by σ^i). Recalling that the motion and branching mechanisms are independent of the initial configuration and injection measure, using the many-to-one lemma [19], we have

$$\mathbb{E}\bigg[\sum_{\hat{i}\in\tilde{\mathcal{S}}^{N,i}_T} 1_{\{\|\tilde{X}^{i,\hat{i}}\|^*_{[\sigma^i,T]}>r\}}\bigg|\tilde{\mathcal{F}}^N_*\bigg] = e^{\kappa(T-\sigma^i)}\mathbb{E}\big[1_{\{\|\tilde{X}^i\|^*_{[\sigma^i,T]}>r\}}|\tilde{\mathcal{F}}^N_*\big], \quad i\in\mathcal{R}^N_T. \tag{4.9}$$

Let X_t solve (2.1) for $t\in[0,T]$, and denote the stochastic integral term in that equation by $C_t = \int_0^t \mathfrak{c}(X_\theta)\,dB_\theta$. Then $\langle C\rangle = \int_0^\cdot \mathfrak{c}(X_s)^2\,ds$, and by time change for continuous martingales,

$$C_t = \check{B}_{\langle C\rangle_t} \tag{4.10}$$

where $\check{B}_s = C_{\tau(s)}$ is a BM, and $\tau(s) = \inf\{t\ge 0: \langle C_t\rangle > s\}$. By the boundedness of the coefficients $\mathfrak{b}$, $\mathfrak{c}$, this gives $\|X-x\|^*_T \le c_1(1+\|\check{B}\|^*_{c_1})$, where c_1 depends only on T and the coefficients. Let $\check{B}^i$ denote the BM, constructed via time change as above, corresponding to $\tilde{X}^i$, $i\in\mathcal{R}^N_T$, and note that for such i, one has that $\tilde{x}^i = x^i$. Then we have shown the inequality $\|\tilde{X}^i - x^i\|^*_{[\sigma^i,T]} \le c_1(1+\|\check{B}^i\|^*_{c_1})$. (Note that the BM $\check{B}^i$ are not in general mutually independent.) Hence, given n, there exists $r' = r'_n$ such that

$$\begin{aligned}\mathbb{P}\bigg(\|\tilde{X}^i - x^i\|^*_{[\sigma^i,T]} > \frac{r'}{2}\bigg|\tilde{\mathcal{F}}^N_0\bigg) &\le \mathbb{P}\bigg(c_1(1+\|\check{B}^1\|^*_{c_1}) > \frac{r'}{2}\bigg)\\ &\le \big(2n(1+c_I)e^{\kappa T}\big)^{-1}2^{-n}, \quad i\in\mathcal{R}^N_T.\end{aligned} \tag{4.11}$$

Next, by our assumptions, the normalized configuration measure of $\{x^i : i \in \mathcal{R}_T^N\}$, given by $\bar{\xi}_0^N + \int_{[0,T]} \bar{\alpha}^N(\cdot, dt)$, converges in probability to a deterministic finite measure on $\mathbb{R}$. Hence, for every n, there exists $r'' = r''_n$ such that

$$\lim_N \mathbb{P}\biggl(\#\biggl\{i \in \mathcal{R}_T^N : |x^i| > \frac{r''}{2}\biggr\} < N(2e^{\kappa T})^{-1} 2^{-n}\biggr) = 1.$$

Hence, recalling $R_T^N \leq (1 + c_I)N$,

$$\limsup_N \mathbb{E}\biggl[\frac{\#\{i \in \mathcal{R}_T^N : |x^i| > r''/2\}}{N}\biggr] \leq (2ne^{\kappa T})^{-1} 2^{-n}. \tag{4.12}$$

For $n \in \mathbb{N}$, let $r_n = r'_n \vee r''_n$. Then, by (4.8), (4.9), (4.11) and finally (4.12),

$$\begin{aligned}
&\limsup_N \mathbb{P}(\bar{\beta}^N(\mathbb{B}_{r_n}^c) \geq n^{-1}) \\
&\leq \limsup_N \frac{n}{N} \mathbb{E}[\tilde{U}_{r_n}^N] \\
&\leq \limsup_N \frac{n}{N} \mathbb{E} \sum_{i \in \mathcal{R}_T^N} \sum_{\hat{i} \in \tilde{\mathcal{S}}_T^{N,i}} 1_{\{\|\tilde{X}^{i,\hat{i}}\|^*_{[\sigma^i, T]} > r_n\}} \\
&\leq \limsup_N \frac{ne^{\kappa T}}{N} \mathbb{E} \sum_{i \in \mathcal{R}_T^N} (1_{\{\|\tilde{X}^i - x^i\|^*_{[\sigma^i, T]} > r_n/2\}} + 1_{\{|x^i| > r_n/2\}}) \\
&\leq 2^{-n}.
\end{aligned} \tag{4.13}$$

This shows (4.6), hence the tightness of the laws of $\bar{\beta}^N$.

Denote by d_{L} the Levy–Prohorov metric on $\mathcal{M}_+(\mathbb{R})$, which is compatible with weak convergence on $\mathcal{M}_+(\mathbb{R})$. We will use the notation $w_T(\cdot, \cdot)$ for both $(\mathbb{R}, |\cdot|)$ and $(\mathcal{M}_+(\mathbb{R}), d_{\mathrm{L}})$. The argument for $\bar{\xi}^N$ is based on showing (i) for every $\varepsilon > 0$ there exists a compact set $K \subset \mathcal{M}_+(\mathbb{R})$ such that $\liminf_N \inf_{t \in [0,T]} \mathbb{P}(\bar{\xi}_t^N \in K) > 1 - \varepsilon$; and (ii) for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_N \mathbb{P}(w_T(\bar{\xi}^N, \delta) > \varepsilon) < \varepsilon.$$

Once these two properties are proved, it will follow that $\bar{\xi}^N$ is a relatively compact sequence [17], Corollary 3.7.4 (page 129). Because we use w rather than w' [17], (3.6.2) (page 122), this will in fact establish C-tightness, proving the second statement.

To show (i), let $c_2 = \|m\|_T^* + 1$. By Lemma 4.1(iii), $\|\bar{m}^N\|_T^* = \sup_{t \in [0,T]} |\bar{\xi}_t^N| \leq c_2$ w.h.p. Denoting

$$\hat{K}_n(r) = \{\gamma \in \mathcal{M}_+(\mathbb{R}) : |\gamma| \leq c_2, \gamma([-r, r]^c) < n^{-1}\},$$

it suffices to prove that for every n there exists r_n such that

$$\liminf_N \inf_t \mathbb{P}(\bar{\xi}_t^N \in \hat{K}_n(r_n)) \geq 1 - 2^{-n},$$

for the same reason given above for (4.6) to be sufficient for tightness of $\bar{\beta}^N$. Moreover, similar to the estimate (4.7) for β^N, we have $\xi_t^N([-r, r]^c) \leq U_r^N$ for all $t \in [0, T]$. Hence, the chain of inequalities (4.13) provides a bound also on $\limsup_N \sup_t \mathbb{P}(\bar{\xi}_t^N \in \hat{K}_n(r_n)^c)$, and (i) follows.

It remains to show (ii). Let $A_t^N = \{i \in \mathcal{S} : \sigma^i \leq t < \tau^i\}$. This is the index set for living particles at time t. For $\delta > 0$, let $a_\delta = \{(s, t) \in [0, T] : 0 < t - s \leq \delta\}$. For $(s, \delta) \in a_\delta$, the number of particles removed during $(s,t]$ is $J_t^{\#,N} - J_s^{\#,N}$. The number of new particles during this interval is given by $I_t^N - I_s^N + Y_t^N - Y_s^N$. Hence, denoting symmetric difference by $\triangle$,

$$\#(A_s^N \triangle A_t^N) \le J_t^{\#,N} - J_s^{\#,N} + I_t^N - I_s^N + Y_t^N - Y_s^N.$$

The convergence of $\bar{I}^N$, $\bar{J}^{\#,N}$ and $\bar{Y}^N$ to continuous paths shows that given $\varepsilon > 0$ there exists $\delta > 0$ such that, w.h.p., for all $(s,t) \in a_\delta$ one has $\#(A_s^N \triangle A_t^N) \le \varepsilon N/2$.

Next, going back to (4.10) and the notation $\check{B}^i$, there exists a constant $c_3 \in (0,\infty)$ such that $w_{[\sigma^i,T]}(X^i,\delta) \le c_3\delta + w_{[\sigma^i,c_3T]}(\check{B}^i, c_3\delta)$. Thus, if $p(\varepsilon,\delta) = \mathbb{P}(c_3\delta + w_{c_3T}(B,c_3\delta) \ge \varepsilon)$ for B a BM, then for $i \in \mathcal{S}$,

$$\mathbb{P}\big(w_{[\sigma^i,T]}\big(X^i,\delta\big) \ge \varepsilon | \mathcal{F}^N_{\sigma^i}\big) \le p(\varepsilon,\delta) \quad \text{on } \{\sigma^i \le T\}.$$

Hence,

$$\begin{aligned}\mathbb{E}\big[\#\{i \in \mathcal{S}_T^N : w_{[\sigma^i,T]}\big(X^i,\delta\big) \ge \varepsilon\}\big] &= \sum_{i\in\mathcal{S}} \mathbb{E}\big[1_{\{\sigma^i \le T\}} \mathbb{E}\big[1_{\{w_{[\sigma^i,T]}(X^i,\delta)\ge\varepsilon\}} | \mathcal{F}^N_{\sigma^i}\big]\big] \\ &\le \mathbb{E}\big[\#\mathcal{S}_T^N\big] p(\varepsilon,\delta) \\ &\le cNp(\varepsilon,\delta),\end{aligned}$$

where in the last inequality we used the fact that the expected number of descendants each root particle has by time T is bounded, which along with the truncation convention of I^N gives $\mathbb{E}[\#\mathcal{S}_T^N] \le cN$. This gives

$$\mathbb{P}\big(\#\{i \in \mathcal{S}_T^N : w_{[\sigma^i,T]}\big(X^i,\delta\big) \ge \varepsilon\} > \varepsilon N/2\big) \le \frac{cNp(\varepsilon,\delta)}{\varepsilon N/2} < \frac{\varepsilon}{4},$$

where we used the fact that $p(\varepsilon, 0+) = 0$ and chose $\delta = \delta(\varepsilon)$ sufficiently small.

Given a set $C \subset \mathbb{R}$, let C^ε denote its ε-neighborhood. We have shown the following. For all N so large that

$$\mathbb{P}\big(\text{for some } (s,t) \in a_\delta, A_s^N \triangle A_t^N > \varepsilon N/2\big) < \frac{\varepsilon}{4},$$

with probability greater than $1 - \varepsilon/2$, for all $(s,t) \in a_\delta$, except for at most $\varepsilon N/2$ particles (removed between s and t), and at most $\varepsilon N/2$ particles (whose displacement exceeds ε), each particle $i \in A_s^N$ exists in the configuration at time t and travels less than ε between s and t. Hence, with probability greater than $1 - \varepsilon/2$, for any Borel set C,

$$\xi_s^N(C) \le \xi_t^N\big(C^\varepsilon\big) + \varepsilon N, \quad (s,t) \in a_\delta.$$

Similarly, with probability greater than $1 - \varepsilon/2$,

$$\xi_t^N(C) \le \xi_s^N\big(C^\varepsilon\big) + \varepsilon N, \quad (s,t) \in a_\delta.$$

Hence, with probability $\ge 1 - \varepsilon$,

$$d_{\mathrm{L}}\big(\bar{\xi}_s^N, \bar{\xi}_t^N\big) \le \varepsilon, \quad (s,t) \in a_\delta.$$

This shows that

$$\limsup_N \mathbb{P}\big(w_T\big(\bar{\xi}^N,\delta\big) > \varepsilon\big) \le \varepsilon,$$

and the proof is complete. □

LEMMA 4.3. *Let (ξ,β) be a subsequential limit of $(\bar{\xi}^N, \bar{\beta}^N)$. Then there exists an event $\Omega_1 \in \mathcal{F}$ of full measure and a $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$-measurable function $u(x,t,\omega)$ such that for every $(t,\omega) \in (0,\infty) \times \Omega_1$, $u(\cdot,t,\omega)$ is a density of $\xi_t(\cdot,\omega)$ with respect to the Lebesgue measure on $\mathbb{R}$. Moreover, for almost all ω, one has*

$$u(x,t,\omega) \le v(x,t) \quad \textit{for all } (x,t) \in \mathbb{R} \times (0,\infty).$$

PROOF. First, we prove that all limits of $\bar{\zeta}^N$ have density. To be more precise, applying Lemma 4.2 with the data $J^N = 0$, in which case $\beta^N = 0$ and $\bar{\xi}^N = \bar{\zeta}^N$, shows that $\bar{\zeta}^N$ are tight. Fix a convergent subsequence and denote its limit by ζ. Recall that

$$v(y,t) = S_t\xi_0(y) + S * \alpha(y,t) = \int_{\mathbb{R}} \mathfrak{s}_t(x,y)\xi_0(dx) + \int_{\mathbb{R}\times[0,t]} \mathfrak{s}_{t-s}(x,y)\alpha(dx,ds),$$

and set $\zeta_t^0(dx) = v(x,t)\,dx$, $t > 0$, and $\zeta_0^0(dx) = \xi_0(dx)$. Then it is a standard result that $\zeta = \zeta^0$ a.s. For completeness, we will prove that, a.s., for all t, $\zeta_t \sqsubset \zeta_t^0$, which is a weaker assertion but is sufficient for our purpose here.

To this end, let $\mathcal{Q}$ denote the set of bounded open intervals $(a,b) \subset \mathbb{R}$. Fix $T > 0$. In the first step, we show that for $t \in (0,T]$, $Q \in \mathcal{Q}$ and $\gamma > 0$,

$$\mathbb{P}\big(\bar{\zeta}_t^N(Q) > \zeta_t^0(Q) + \gamma\big) \to 0. \tag{4.14}$$

For a first moment calculation, using the many to one lemma as before,

$$\begin{aligned}\mathbb{E}[\bar{\zeta}_t^N(Q)|\tilde{\mathcal{F}}_*^N] &= N^{-1}\mathbb{E}\Big[\sum_{i\in\mathcal{R}_t^N}\sum_{\hat{i}\in\mathcal{S}_t^{N,i}} 1_{\{\tilde{X}_t^{i,\hat{i}}\in Q\}}\Big|\tilde{\mathcal{F}}_*^N\Big] \\ &= N^{-1}\sum_{i\in\mathcal{R}_t^N} e^{\kappa(t-\sigma^i)}\mathbb{P}\big(\tilde{X}_t^i \in Q|\tilde{\mathcal{F}}_*^N\big) \\ &= \bar{\Theta}^N := N^{-1}\sum_{i\in\mathcal{R}_t^N}\theta^i(N),\end{aligned}$$

where $\theta^i(N) = \mathfrak{s}_{t-\sigma^i}(x^i, Q)$. Now, for $\varepsilon \in (0,t)$,

$$\begin{aligned}\bar{\Theta}^N &= \int_{\mathbb{R}} \mathfrak{s}_t(x,Q)\bar{\xi}_0^N(dx) + \int_{\mathbb{R}\times[0,t]} \mathfrak{s}_{t-s}(x,Q)\bar{\alpha}^N(dx,ds) \\ &\le V^N(t,\varepsilon,Q) \\ &:= \int_{\mathbb{R}} \mathfrak{s}_t(x,Q)\bar{\xi}_0^N(dx) + \int_{\mathbb{R}\times[0,t-\varepsilon]} \mathfrak{s}_{t-s}(x,Q)\bar{\alpha}^N(dx,ds) \\ &\quad + e^{\kappa T}\big(\bar{I}_t^N - \bar{I}_{t-\varepsilon}^N\big).\end{aligned} \tag{4.15}$$

On $\mathbb{R}\times[0,t-\varepsilon]$, $(x,s) \mapsto \mathfrak{p}_{t-s}(x,Q)$ is bounded and continuous [24], Theorem 1.2.1. Since α does not charge $\mathbb{R}\times\{t-\varepsilon\}$, $V^N(t,\varepsilon,Q)$ converges in probability to

$$V(t,\varepsilon,Q) := \int_{\mathbb{R}} \mathfrak{s}_t(x,Q)\xi_0(dx) + \int_{\mathbb{R}\times[0,t-\varepsilon]} \mathfrak{s}_{t-s}(x,Q)\alpha(dx,ds) + c(I_t - I_{t-\varepsilon}).$$

We may adopt the truncation convention from the proof of Lemma 4.2. Thus, $\bar{I}_T^N$ and $|\bar{\alpha}^N|$ are bounded, and we have by bounded convergence $\mathbb{E}[V^N(t,\varepsilon,Q)] \to V(t,\varepsilon,Q)$, which upon taking $\varepsilon \to 0$, gives

$$\limsup \mathbb{E}[\bar{\Theta}^N] \le \int_Q S_t\xi_0(y)\,dy + \int_Q S * \alpha(y,t)\,dy = \zeta_t^0(Q).$$

Moreover, dropping the last term in (4.15) gives a lower bound on $\bar{\Theta}^N$, hence similarly $\liminf \mathbb{E}[\bar{\Theta}^N] \ge \zeta_t^0(Q)$, giving

$$\lim \mathbb{E}[\bar{\zeta}_t^N(Q)] = \lim \mathbb{E}[\bar{\Theta}^N] = \zeta_t^0(Q). \tag{4.16}$$

Similarly,

$$\limsup \mathbb{E}\big[(\bar{\Theta}^N)^2\big] \le \zeta_t^0(Q)^2. \tag{4.17}$$

By Lemma 3.3, $\zeta_t^0(Q) < \infty$. For a second moment calculation, if $i \in \mathcal{R}_t^N$ then

$$\mathbb{E}\biggl[\sum_{\hat{i}_1,\hat{i}_2 \in \tilde{\mathcal{S}}_t^{N,i}} 1_{\{\tilde{X}_t^{\hat{i}_1} \in Q, \tilde{X}_t^{\hat{i}_2} \in Q\}} \Big| \tilde{\mathcal{F}}_*^N\biggr] \le \mathbb{E}\bigl[(\tilde{Z}_T^{N,i})^2 | \tilde{\mathcal{F}}_*^N\bigr] \le c,$$

whereas if $i_1, i_2 \in \mathcal{R}_T^N$ are distinct, using conditional independence and the many-to-one lemma,

$$\mathbb{E}\biggl[\sum_{\hat{i}_1 \in \tilde{\mathcal{S}}_T^{N,i_1}, \hat{i}_2 \in \tilde{\mathcal{S}}_T^{N,i_2}} 1_{\{\tilde{X}_t^{\hat{i}_1} \in Q, \tilde{X}_t^{\hat{i}_2} \in Q\}} \Big| \tilde{\mathcal{F}}_*^N\biggr] = e^{\kappa(t-\sigma^{i_1}+t-\sigma^{i_2})} \mathbb{P}\bigl(\tilde{X}_t^{i_1} \in Q | \tilde{\mathcal{F}}_*^N\bigr) \mathbb{P}\bigl(\tilde{X}_t^{i_2} \in Q | \tilde{\mathcal{F}}_*^N\bigr).$$

Therefore,

$$\mathbb{E}[\bar{\zeta}_t^N(Q)^2 | \tilde{\mathcal{F}}_*^N] \le cN^{-1} + (\bar{\Theta}^N)^2,$$

and using now (4.17), $\limsup \mathbb{E}[(\bar{\zeta}_t^N(Q))^2] \le \zeta_t^0(Q)^2$. In view of (4.16), this gives that $\lim \operatorname{var}(\bar{\zeta}_t^N(Q)) = 0$. By (4.16), this shows (4.14).

Next, it is shown that for every $t \in (0, T]$ and $Q \in \mathcal{Q}$ there is a full-measure event on which $\zeta_t(Q) \le \zeta_t^0(Q)$. Since along a subsequence one has $\bar{\zeta}^N \Rightarrow \zeta$ and the latter has continuous sample paths, one also has $\bar{\zeta}_t^N \Rightarrow \zeta_t$. Using Skorohod's representation, we may assume w.l.o.g. that $\bar{\zeta}_t^N \to \zeta_t$ a.s., and since Q is open, we have $\liminf \bar{\zeta}_t^N(Q) \ge \zeta_t(Q)$ a.s. Hence, with $\tilde{c} = \zeta_t^0(Q) + \gamma$,

$$\mathbb{P}\bigl(\zeta_t(Q) > \tilde{c}\bigr) \le \mathbb{P}\bigl(\liminf \bar{\zeta}_t^N(Q) > \tilde{c}\bigr) \le \mathbb{E} \liminf 1_{\{\bar{\zeta}_t^N(Q) > \tilde{c}\}} \le \liminf \mathbb{P}\bigl(\bar{\zeta}_t^N(Q) > \tilde{c}\bigr) = 0,$$

where we have used the lower semicontinuity of $x \mapsto 1_{\{x > \tilde{c}\}}$ and Fatou's lemma. This shows $\mathbb{P}(\zeta_t(Q) > \zeta_t^0(Q) + \gamma) = 0$, and γ can now be dropped.

Let $\tilde{\mathcal{Q}} \subset \mathcal{Q}$ be the set of open intervals (a, b) with $a, b \in \mathbb{Q}$. Then there exists an event Ω_0 of full measure on which for all $t \in (0, T] \cap \mathbb{Q}$ and all $Q \in \tilde{\mathcal{Q}}$, $\zeta_t(Q) \le \zeta_t^0(Q)$ holds. Using the continuity of $t \mapsto \xi_t$, we have that, on Ω_0, the same holds for all $(t, Q) \in (0, T] \times \tilde{\mathcal{Q}}$. The last assertion can be extended to $(0, T] \times \mathcal{Q}$ by taking $\tilde{\mathcal{Q}} \ni Q_n \uparrow Q \in \mathcal{Q}$. It follows (e.g., [5], Corollary 2, page 169) that on an event of full measure,

$$\zeta_t \sqsubset \zeta_t^0, \quad t \in (0, T]. \tag{4.18}$$

Recall that $\bar{\xi}_t^N \sqsubset \bar{\zeta}_t^N$ for all t. Then every subsequential limit ξ of $\bar{\xi}^N$ must also satisfy, on an event of full measure, $\xi_t \sqsubset \zeta_t^0$ for all $t \in (0, T]$. In particular, $\xi_t(dx) \ll dx$. Since T is arbitrary, this holds on $(0, \infty)$, and setting $\xi = 0$ outside the full measure event, we finally obtain that for every $(t, \omega) \in (0, \infty) \times \Omega$, $\xi_t(dx, \omega) \ll dx$. We now appeal to [15], Theorem 58 in Chapter V (page 52), and the remark that follows. The measurable spaces denoted in [15] by $(\Omega, \mathcal{F})$ and $(T, \mathcal{T})$ are taken to be $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $((0, \infty) \times \Omega, \mathcal{B}((0, \infty)) \otimes \mathcal{F})$, respectively. According to this result, there exists a $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}((0, \infty)) \otimes \mathcal{F}$-measurable function $u(x, t, \omega)$, such that for every $(t, \omega) \in (0, \infty) \times \Omega$, $u(\cdot, t, \omega)$ is a density of $\xi_t(dx, \omega)$ with respect to dx.

For the last assertion of the lemma, note that by (4.18), modifying u into $u \wedge v$ still gives a density of $\xi_t(\cdot, \omega)$. □

4.2. *Limit laws and the complementarity condition.* Here, we prove that the complementarity condition carries over to the limit.

LEMMA 4.4. (i) *Let* $u \in L_{1,\mathrm{loc}}(\mathbb{R}_+, L_1) \cap L_{\infty,\mathrm{loc}}((0, \infty), L_\infty)$ *be positive. Denote* $U(x, t) = \int_{-\infty}^x u(y, t)\, dy$. *Let* $\beta \in \mathcal{M}_{+,\mathrm{loc}}(\mathbb{R} \times \mathbb{R}_+)$, $\beta(\mathbb{R} \times \{0\}) = 0$. *Then the following* *two conditions are equivalent*:

$$\beta(U > 0) = 0, \tag{4.19}$$

$$\mathcal{I}_r := \int_{[r,\infty)\times\mathbb{R}_+} U(r,t)\beta(dx, dt) = 0 \quad \textit{for all } r \in \mathbb{R}. \tag{4.20}$$

(ii) *Let* (ξ, β) *be a subsequential limit of* $(\bar{\xi}^N, \bar{\beta}^N)$. *Then a.s.*,

$$\mathcal{I}_r(\xi, \beta) := \int_{[r,\infty)\times\mathbb{R}_+} \xi_t(-\infty, r]\beta(dx, dt) = 0 \quad \textit{for all } r \in \mathbb{R}. \tag{4.21}$$

(iii) *Consequently, if* u *is the density from Lemma* 4.3 (*defined arbitrarily on* $\mathbb{R} \times \{0\}$) *and* $U(x, t, \omega) = \int_{-\infty}^{x} u(y, t, \omega)\, dy$ *then* (U, β) *satisfy* (4.19) *a.s.*

PROOF. (i) To show that (4.19) implies (4.20), write

$$\mathcal{I}_r \le \int_{[r,\infty)\times\mathbb{R}_+} U(x,t)\beta(dx, dt) \le \int_{\mathbb{R}\times\mathbb{R}_+} U(x,t)\beta(dx, dt) = 0.$$

For the converse, assume (4.19) is false. Then there exists $\delta > 0$, $\beta(U > \delta) > 0$. Since β does not charge the set $\mathbb{R} \times \{0\}$, there exist $0 < t_1 < t_2 < \infty$ and finite $a < b$ such that, with $K = [a, b] \times [t_1, t_2]$, $\beta(K \cap \{U > \delta\}) > 0$. With c an upper bound on u in K, and $\varepsilon = \frac{\delta}{2c} \wedge (b - a)$,

$$U(y, t) - U(x, t) \le \frac{\delta}{2} \quad x, y \in [a, b], 0 \le y - x \le \varepsilon, t \in [t_1, t_2].$$

Moreover, there exists $r \in [a, b]$ such that $\beta(L) > 0$ where $L = [r, r + \varepsilon] \times [t_1, t_2] \cap \{U > \delta\}$. Hence, for $(x, t) \in L$, $U(r, t) \ge U(x, t) - \frac{\delta}{2} \ge \delta - \frac{\delta}{2} = \frac{\delta}{2}$. Thus,

$$\mathcal{I}_r \ge \int_L U(r, t)\beta(dx, dt) \ge \frac{\delta}{2}\beta(L) > 0,$$

showing that (4.20) is false.

(ii) Let Σ be the collection of tuples $\sigma = (r, t_1, t_2, \eta, \delta) \in \mathbb{Q}^5$, $0 < t_1 < t_2$, $\eta > 0$, $\delta > 0$. We show that for every $\sigma \in \Sigma$, $\mathbb{P}(\Omega_\sigma) = 0$ where

$$\Omega_\sigma = \Big\{\inf_{t\in[t_1,t_2]} \xi_t(-\infty, r] > \eta, \beta\big((r, \infty) \times (t_1, t_2)\big) > \delta\Big\}.$$

Fix $\sigma = (r, t_1, t_2, \eta, \delta)$. If $\mathbb{P}(\Omega_\sigma) > 0$, then by the weak convergence $(\bar{\xi}^N, \bar{\beta}^N) \Rightarrow (\xi, \beta)$, the a.s. continuity of the limit ξ, and the fact that, for every $t > 0$, the measure ξ_t has no atoms, one must have for all large N,

$$\mathbb{P}\Big(\inf_{t\in[t_1,t_2]} \bar{\xi}_t^N(-\infty, r] > \eta/2, \bar{\beta}^N(r, \infty) \times (t_1, t_2) > \delta/2\Big) > 0.$$

However, by the construction of the particle system, for every r, a removal never occurs at a location $> r$ at a time when there are particles at location $\le r$. Hence, the above probability is zero for all N. This shows $\mathbb{P}(\Omega_\sigma) = 0$. Consequently, $\mathbb{P}(\bigcup_\Sigma \Omega_\sigma) = 0$.

Next, consider the event

$$\Omega^0 = \{\text{there exists } r \in \mathbb{R} \text{ such that } \mathcal{I}_r(\xi, \beta) > 0\}.$$

On this event, there exists $r \in \mathbb{R}$ and $0 < s_1 < s_2 < \infty$ such that

$$\mathcal{I}^* := \int_{[r,\infty)\times(s_1,s_2)} \xi_t(-\infty, r]\beta(dx, dt) > 0.$$

Consider a sequence $\mathbb{Q} \ni r_n \uparrow r$. Recalling that the density $u(\cdot,\cdot,\omega)$ is bounded by v and denoting $\gamma_1 = \sup_{(x,t)\in[r-1,r]\times(s_1,s_2)} v(x,t)$, $\gamma_2 = \beta(\mathbb{R}\times(s_1,s_2))$,

$$\begin{aligned}\mathcal{I}_{r_n}(\xi,\beta) &\geq \int_{[r_n,\infty)\times(s_1,s_2)} \xi_t(-\infty,r_n]\beta(dx,dt)\\ &\geq \int_{[r,\infty)\times(s_1,s_2)} \big(\xi_t(-\infty,r]-\xi_t(r_n,r]\big)\beta(dx,dt)\\ &\geq \mathcal{I}^* - \sup_{t\in(s_1,s_2)} \xi_t(r_n,r]\gamma_2\\ &\geq \mathcal{I}^* - (r-r_n)\gamma_1\gamma_2 > 0\end{aligned}$$

for large n. This shows that on Ω^0 there exists $r\in\mathbb{Q}$ such that $\mathcal{I}_r(\xi,\beta)>0$.

Next, the condition $\mathcal{I}_r(\xi,\beta)>0$ (with $r\in\mathbb{Q}$) implies that there exists $\eta\in\mathbb{Q}\cap(0,1)$ such that $\int_{A_\eta} a_t\,db_t > 0$ where we denote $a_t = \xi_t(-\infty,r] = \xi_t(-\infty,r)$, $b_t = \beta(r,\infty)\times[0,t]$ and $A_\eta = \{t : a_t > 2\eta\}$. The trajectory $t\mapsto a_t$ is continuous on $(0,\infty)$ (using the fact that $\xi\in C(\mathbb{R}_+,\mathcal{M}_+(\mathbb{R}))$ and that for each t, ξ_t has no atoms). Hence, there exists an interval $(t_1,t_2)\subset A_\eta$, with $t_1,t_2\in\mathbb{Q}$, such that $\int_{(t_1,t_2)} a_t\,db_t>0$. Consequently,

$$\xi_t(-\infty,r] = a_t \geq 2\eta > \eta$$

on $[t_1,t_2]$, while

$$\beta\big((r,\infty)\times(t_1,t_2)\big) = \int_{(t_1,t_2)} db_t > 0.$$

This shows that $\mathbb{P}(\Omega^0) \leq \mathbb{P}(\bigcup_{\sigma\in\Sigma}\Omega_\sigma) = 0$.

(iii) The final assertion follows from the first two as soon as these conditions are verified: $\|u(\cdot,t,\omega)\|_\infty$ is locally bounded for $t\in(0,\infty)$, and $\beta(\mathbb{R}\times\{0\})=0$. The former follows from Lemmas 4.3 and 3.3(iii), by which $u(\cdot,\cdot,\omega)\leq v$ and $v\in L_{\mathrm{loc},\infty}((0,\infty),L_\infty)$. The latter follows from Lemma 4.1(iii) by which $\bar{J}^{\#,N}\to J$ and the assumption $J_0=0$. □

4.3. *Proof of Theorem* 2.5. In view of the tightness stated in Lemma 4.2 and the uniqueness of solutions to (2.10) stated in Theorem 3.1, it suffices to show that whenever $(\xi_0,\xi,\alpha,\beta,J)$ is a subsequential limit of $(\bar{\xi}_0^N,\bar{\xi}^N,\bar{\alpha}^N,\bar{\beta}^N,\bar{J}^N)$, and u the corresponding density from Lemma 4.3, one has that, a.s., (u,β) is a solution to (2.10).

That $u\in L_{1,\mathrm{loc}}(\mathbb{R}_+,L_q)$ for $q\in(1,\infty)$ follows from Lemma 4.3, which states that $u\leq v$, and Lemma 3.3(i), by which $v\in L_{1,\mathrm{loc}}(\mathbb{R}_+,L_q)$ for all $q\in(1,\infty)$. Since by Lemma 4.1(iii) $\bar{J}^{\#,N}\to J$, we have $\beta(\mathbb{R}\times[0,t])=J_t<\infty$ for all t, showing that $\beta\in\mathcal{M}_{+,\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+)$. Thus, to show that u is a weak L_q-solution to (2.10)(i), it remains to show that (2.11) holds. This is indeed the case by Lemma 4.1(iv), in view of the relation $\xi_t(dx)=u(x,t)\,dx$.

Finally, Lemma 4.4 shows that condition (2.10)(ii) holds, and for condition (2.10)(iii) we have just provided a proof. This shows that, a.s., (u,β) is a solution to (2.10), and completes the proof.

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