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A Durrett–Remenik particle system in $\mathbb{R}^d$

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ABSTRACT

This paper studies a branching-selection model of motionless particles in $\mathbb{R}^d$, with nonlocal branching, introduced by Durrett and Remenik in dimension 1. The assumptions on the fitness function, F, and on the inhomogeneous branching distribution, are mild. The evolution equation for the macroscopic density is given by an integro-differential free boundary problem in $\mathbb{R}^d$, in which the free boundary represents the least F-value in the population. The main result is the characterization of the limit in probability of the empirical measure process in terms of the unique solution to this free boundary problem.

1. Introduction

A system of particles living in $\mathbb{R}^d$, $d \geq 1$, that undergo branching and selection, is described as follows. An initial configuration is given by $\{x^i, 1 \leq i \leq N\}$, where x^i are $\mathbb{R}^d$-valued random variables. An independent Poisson clock of rate 1 is attached to each particle, according to which it gives birth to new ones. The location of a particle born from one located at y is drawn according to a probability density $\rho(y, \cdot)$ defined on $\mathbb{R}^d$. All particles stay put throughout their lifetime. When a particle is born, the particle that has the least F-value among the living ones (including the newborn) is removed, where $F : \mathbb{R}^d \to \mathbb{R}$ is a given 'fitness' function. Tie breaking (which may be needed for the initial particles) occurs according to particle labels. Note that the number of particles remains N at all times. The label of the particle that gets removed is transferred to the new particle, so that at all times, the living particles are labeled as $1, 2, \ldots, N$. The location of particle i at time t is denoted by $X_i(t)$ (suppressing the dependence on N), the configuration measure by $\xi_t^N = \sum \delta_{X_i(t)}$, and the empirical measure by $\bar{\xi}_t^N = N^{-1}\xi_t^N$. This model was introduced and studied in [1] for $d = 1$ and $F(x) = x$, where its hydrodynamic limit was characterized in terms of an integro-differential free boundary problem (FBP).

This result is extended in this paper. Under mild assumptions on ρ and F, it is shown that the empirical measure process converges in probability to a deterministic path in measure space, whose density uniquely solves an integro-differential FBP in $\mathbb{R}^d$. This is the problem of finding a pair (u, ℓ), where $\ell : \mathbb{R}_+ \to \mathbb{R}$ is càdlàg and the function $u : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}_+$ is $C^{0,1}$ in $\{(x,t) \in \mathbb{R} \times \mathbb{R}_+ : F(x) > \ell_t\}$, and one has

$$\begin{cases} \partial_t u(x,t) = \displaystyle\int_{\mathbb{R}^d} u(y,t)\rho(y,x)dy & (x,t) : F(x) > \ell_t,\ t > 0, \\ u(x,t) = 0 & (x,t) : F(x) \leq \ell_t,\ t > 0, \\ \displaystyle\int_{\mathbb{R}^d} u(x,t)dx = 1, & t > 0, \\ u(\cdot,0) = u_0,\ \ell_0 = \lambda_0. \end{cases}$$

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The function u and the free boundary ℓ represent the density and, respectively, the least F-value in the population at the macroscopic scale, with u_0 and λ_0 their initial conditions.

The relation between particle system with selection and FBP is an active research field. Particle systems with selection were proposed in [2,3] as models for natural selection in population dynamics, where the position of a particle on the real line represents the degree of fitness of an individual to its environment, and those that are least fitted get removed. In [1], the aforementioned Durrett–Remenik model for motionless particles was introduced, where, in addition to hydrodynamic limits, traveling wave solutions were also studied. The monograph [4] studied hydrodynamic limits of Brownian particle systems with injection and selection, relating them to FBP. A model for Brownian particles with branching and selection, referred to as N-particle branching Brownian motion (N-BBM), related to the ones proposed in [2,3], was introduced in [5], and studied at the hydrodynamic limit in [6,7]. A variant of N-BBM with nonlocal branching and its relation to FBP was studied in [8]. Other hydrodynamic limit results for branching-selection models were studied in [9,10]. Apart from selection models, particle systems that give rise to FBP include a simple exclusion process with boundary behavior [11] and the Atlas model [12,13].

The works mentioned above all address FBP in dimension 1. In higher dimension, [14] studied the asymptotic shape of the cloud of particles of the N-BBM with fitness function $F(x) = \|x\|$ and $F(x) = \lambda \cdot x$. The papers [15,16] considered the hydrodynamic limit of the N-BBM with $F(x) = -\|x\|$, characterizing it in terms of a FBP, studying convergence rates as well as long time behavior. This treatment used in a crucial way the symmetry of F, by which the radial projection of the macroscopic dynamics is governed by an autonomous equation, and thus the motion of the free boundary is dictated by an equation in one dimension.

Our motivation in this work is to study a branching-selection model for which the macroscopic evolution equation is truly multidimensional, in the sense that it does not reduce to a FBP in one spatial dimension. This is relevant also from the application viewpoint. For example, in the context of natural selection, consider scores given to different traits, each represented by one of the coordinates x_i of the state x of an individual. In this case, the fitness functions $F(x) = \min_i x_i$ and $F(x) = \max_i x_i$ may be of interest, as they represent scenarios where the worst (resp., best) score among the various traits governs extinction.

The Durrett–Remenik and N-BBM models in $\mathbb{R}^d$ are both multidimensional branching-selection models, and so they share some similarities, but there are significant differences as far as their analysis is concerned. The most obvious one is an integro-differential FBP studied here for the former versus a second order parabolic FBP studied in [15,16] for the N-BBM. The tools required are different, where, for example, the very question of existence of classical solutions to the PDE of [15,16] is difficult. Their results not only obtain the hydrodynamic limit but also large time asymptotics, which we do not address here. In this paper, FBP uniqueness builds on monotonicity of the free boundary ℓ, whereas this property does not hold in general for N-BBM. Finally, for the N-BBM, establishing the hydrodynamic limit for general fitness functions under which the dynamics do not reduce to one dimension is a challenging open question.

The rest of this article is organized as follows. In Section 2, assumptions are introduced, the main result is stated, and the proof is outlined. The proof is given in Section 3, starting with Section 3.1 where uniqueness of solutions to the FBP is proved. In Section 3.2, it is shown that the empirical measure processes form a tight sequence and that limits are supported on solutions to the FBP, which completes the proof. A construction required for one of the steps in Section 3.2 is provided in the Appendix.

Notation. Denote $\mathbb{R}_+ = [0,\infty)$. Let $\mathcal{B}(\mathbb{R}^d)$ denote the class of Borel subsets of $\mathbb{R}^d$. In $\mathbb{R}^d$, denote the Euclidean norm by $\|\cdot\|$ and let $\mathbb{B}_r = \{x \in \mathbb{R}^d : \|x\| \le r\}$. Denote by $\mathcal{M}(\mathbb{R}^d)$ the space of finite signed Borel measures on $\mathbb{R}^d$ endowed with the topology of weak convergence. Let $\mathcal{P}(\mathbb{R}^d) \subset \mathcal{M}_+(\mathbb{R}^d) \subset \mathcal{M}(\mathbb{R}^d)$ denote the subsets of probability and, respectively positive measures, and give them the inherited topologies. For $\mu, \nu \in \mathcal{M}_+(\mathbb{R}^d)$, write $\mu \sqsubset \nu$ if $\mu(A) \le \nu(A)$ for all $A \in \mathcal{B}(\mathbb{R}^d)$.

For (X, d_X) a Polish space let $C(\mathbb{R}_+, X)$ and $D(\mathbb{R}_+, X)$ denote the space of continuous and, respectively, càdlàg paths, endowed with the topology of uniform convergence on compacts and, respectively, the Skorohod J_1 topology. Let $C^{\uparrow}(\mathbb{R}_+, \mathbb{R}_+)$ denote the subset of $C(\mathbb{R}_+, \mathbb{R}_+)$ of nondecreasing functions that vanish at zero. For $f : \mathbb{R}_+ \to \mathbb{R}^d$ denote

$$w_T(f,\delta) = \sup\{\|f(t) - f(s)\| : 0 \le s \le t \le (s+\delta) \wedge T\},$$
$$\|f\|_T^* = \sup\{\|f(s)\| : s \in [0,T]\}.$$

2. Assumptions and main result

2.1. Assumptions and macroscopic dynamics

Recall that the empirical measure is given by $\xi_t^N = N^{-1} \sum \delta_{X_i(t)}$. In particular, the initial empirical measure is $\bar{\xi}_0^N = N^{-1} \sum \delta_{x^i}$. Following is our assumption on F, ρ and $\bar{\xi}_0^N$. Denote $\ell_0^N = \min\{F(y) : y \in \mathrm{supp}(\xi_0^N)\}$.

Assumption 2.1. (i) $F \in C(\mathbb{R}^d, \mathbb{R})$, $\inf_x F(x) = -\infty$, $\sup_x F(x) = \infty$, and for every $a \in \mathbb{R}$, $F^{-1}\{a\}$ has Lebesgue measure zero.

(ii) There exists a probability density $\tilde{\rho}$ and a constant $\tilde{c}$ such that $\rho(x,y) \le \tilde{c}\tilde{\rho}(y-x)$. Moreover, $\rho(x,y)$ is continuous in y uniformly in (x,y), and for $a \in \mathbb{R}$ and $x \in F^{-1}(a,\infty)$, one has $\int_{F^{-1}(a,\infty)} \rho(x,y)dy > 0$.

(iii) As $N \to \infty$, $(\bar{\xi}_0^N, \ell_0^N) \to (\xi_0, \lambda_0)$ in probability, where the latter tuple is deterministic, $\xi_0(dx) = u_0(x)dx$ and $\lambda_0 \in \mathbb{R}$. Moreover, u_0 is bounded and continuous on $F^{-1}(\lambda_0,\infty)$ (and necessarily vanishes on $F^{-1}(-\infty,\lambda_0)$), and for every $\delta > 0$ and $\lambda \ge \lambda_0$, $\int_{F^{-1}(\lambda,\lambda+\delta)} u_0(x)dx > 0$.

Further notation is $\ell_t^N = \min\{F(y) : y \in \mathrm{supp}(\xi_t^N)\}$ for the minimal F-value of all living particles at time t, and J^N for the removal counting process. By construction, J^N is a Poisson process of rate N.

Next is an equation for the macroscopic dynamics, in the form of an integro-differential FBP. Denote by $\mathcal{X}$ the set of pairs (u, ℓ), where $\ell \in D(\mathbb{R}_+, \mathbb{R})$ and $u : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}_+$ is $C^{0,1}$ in $\{(x,t) \in \mathbb{R} \times \mathbb{R}_+ : F(x) > \ell_t\}$ and bounded on $\mathbb{R}^d \times [0,T]$ for any T. Consider the system

$$\begin{cases} (i) & \partial_t u(x,t) = \int_{\mathbb{R}^d} u(y,t)\rho(y,x)dy \qquad (x,t) : F(x) > \ell_t,\ t > 0, \\ (ii) & u(x,t) = 0 \qquad (x,t) : F(x) \le \ell_t,\ t > 0, \\ (iii) & \int_{\mathbb{R}^d} u(x,t)dx = 1, \qquad t > 0, \\ (iv) & u(\cdot,0) = u_0,\ \ell_0 = \lambda_0. \end{cases} \tag{2.1}$$

A solution to (2.1) is defined as a member of $\mathcal{X}$ satisfying (2.1).

2.2. Main result

Theorem 2.2. *Let Assumption 2.1 hold. Then there exists a unique solution to (2.1), denoted by (u, ℓ). Moreover, $\ell - \lambda_0 \in C^{\uparrow}(\mathbb{R}_+, \mathbb{R}_+)$. Furthermore, $(\bar{\xi}^N, \ell^N) \to (\xi, \ell)$, in probability, in $D(\mathbb{R}_+, \mathcal{P}(\mathbb{R}^d)) \times D(\mathbb{R}_+, \mathbb{R}_+)$, where $\xi_t(dx) = u(x,t)dx$.*

Remark 2.3. Most parts of Assumption 2.1 are quite mild. The parts that may require some comment are the last part of Assumption 2.1(ii) and the last part of Assumption 2.1(iii). These are used to show that ℓ is monotone whenever (u, ℓ) is a solution of the FBP (see Lemma 3.1), and, respectively, that ℓ^N are C-tight (see Lemma 3.3).

Remark 2.4. Because, as stated above, the ℓ component of any solution to (2.1) is nondecreasing, (2.1)(i) can be written in integral form as

$$u(x,t) = u_0(x) + \int_0^t \int_{\mathbb{R}^d} u(y,s)\rho(y,x)dyds, \qquad (x,t) : F(x) > \ell_t, t > 0.$$

In a more general setting not covered in this paper, where the mass conservation condition is replaced by $\int_{\mathbb{R}^d} u(x,t)dx = m(t)$, with $m(t)$ given, ℓ need not be nondecreasing. In this case the system (2.1) (with 1 replaced by $m(t)$ on the r.h.s. of (iii)) is not sufficient for characterizing (u, ℓ). Roughly speaking, a boundary condition $u(\ell_t, t) = 0$ should be added at times when ℓ_t is decreasing. A precise way to write this is

$$u(x,t) = u_0(x)1_{\{\tau(t,x)=0\}} + \int_{\tau(t,x)}^t \int_{\mathbb{R}^d} u(y,s)\rho(y,x)dyds, \qquad (x,t) : F(x) > \ell_t, t > 0,$$

where $\tau(x,t) = \inf\{s \in [0,t] : \ell_\theta < x \text{ for all } \theta \in (s,t)\}$.

Remark 2.5. Our proof corrects a mistake in the proof of uniqueness of solutions to the integro-differential FBP in [1] (this is equation (FB), corresponding to (2.1) above in the case $d = 1$ and $F(x) = x$; see [17, Remark 2.10] for details about the mistake). Addressing uniqueness via a different approach, our treatment validates the uniqueness statement of [1].

2.3. Proof outline

The proof of Theorem 2.2 is based on the compactness–uniqueness approach, and proceeds in three main steps.

(a) *Uniqueness.* This is the content of Section 3.1, where it is shown that (2.1) has at most one solution.

For a quick sketch of the idea, consider the case $d = 1$ and $F(x) = x$. If (u, ℓ) and (v, m) are solutions then, for each $t > 0$, $u(x,t)$ vanishes for $x < \ell_t$ and $v(x,t)$ vanishes for $x < m_t$. This and the fact that u and v have the same mass imply

$$\int_{\mathbb{R}} |u(x,t) - v(x,t)|dx \le 2\int_{\ell_t \vee m_t}^{\infty} |u(x,t) - v(x,t)|dx.$$

For both u and v, the r.h.s. now involves only (x,t) for which the integro-differential equation (2.1)(i) holds. Integrating it over time allows the use of Gronwall's lemma.

(b) *Tightness.* In particular, in Lemma 3.3 of Section 3.2, it is shown that $(\bar{\xi}^N, \ell^N)$ forms a C-tight sequence.

(c) *Subsequential limits form FBP solutions.* This is shown in the remainder of Section 3.2.

The idea here is to use the standard fact that normalized configuration measures of particle systems with fixed boundary converge as $N \to \infty$ to solutions to integro-differential equations with fixed boundary.

To exploit this fact, upper and lower piecewise constant (but random) envelopes of the prelimit free boundary ℓ^N are selected. Then, auxiliary particle systems, with piecewise constant boundaries given by these envelopes, are constructed. It is argued that the configuration measures of the auxiliary particle systems form lower and upper bounds on ξ_t^N, in the sense of measure inequalities. The aforementioned convergence as $N \to \infty$ can then be applied in order to sandwich any subsequential limit of $\bar{\xi}_t^N$ in terms of solutions to the corresponding integro-differential equations. Because ℓ^N are C-tight, the upper and lower envelopes can be made arbitrarily close to each other locally uniformly. A lemma showing that the solutions of the auxiliary integro-differential equations are continuous with respect to perturbations in the boundary then completes the argument.

3. Proof of main result

3.1. Uniqueness

Before implementing the argument outlined in Section 2.3 part (a), we need the following.

Lemma 3.1. *Let $(u,\ell)\in\mathcal{X}$ be a solution of (2.1). Then ℓ is nondecreasing.*

Proof. Arguing by contradiction, assume there exists $t>0$ such that $\ell_t < L_t := \sup_{s\in[0,t]}\ell_s$. There are two possibilities.

1. There is $s<t$ such that $\ell_s = L_t$. In this case, $\ell_\theta \le \ell_s$ for all $\theta\in[s,t]$. If for some x $F(x)>\ell_s$ then $F(x)>\ell_\theta$ for all $\theta\in[s,t]$, and we can integrate (2.1)(i). Thus

$$u(x,t) = u(x,s) + \int_s^t \int_{\mathbb{R}^d} u(y,\theta)\rho(y,x)dyd\theta, \qquad x\in F^{-1}(\ell_s,\infty). \tag{3.1}$$

We obtain

$$\begin{aligned} 1 = \int_{\mathbb{R}^d} u(x,t)dx &\ge \int_{F^{-1}(\ell_s,\infty)} \Big[u(x,s) + \int_s^t \int_{\mathbb{R}^d} u(y,\theta)\rho(y,x)dyd\theta\Big]dx \\ &\ge 1 + \int_s^t \int_{F^{-1}(\ell_s,\infty)} \int_{F^{-1}(\ell_s,\infty)} u(y,\theta)\rho(y,x)dydxd\theta. \end{aligned}$$

For every $y\in F^{-1}(\ell_s,\infty)$, one has $\psi(y) := \int_{F^{-1}(\ell_s,\infty)}\rho(y,x)dx>0$, by Assumption 2.1(ii). Since $\ell_s\ge\ell_\theta$ for all $\theta\in[0,t]$, we have, similarly to (3.1), that

$$u(y,\theta) = u_0(y) + \int_0^\theta \int_{\mathbb{R}^d} u(y',\theta')\rho(y',y)dy'd\theta', \qquad y\in F^{-1}(\ell_s,\infty).$$

Hence for such y and all $\theta\in[s,t]$ one has $u(y,\theta)\ge u_0(y)$. Hence the triple integral above is bounded below by

$$(t-s)\int_{F^{-1}(\ell_s,\infty)} u_0(y)\psi(y)dy.$$

But $\int_{F^{-1}(\ell_s,\infty)} u_0(y)dy>0$ by assumption, hence the above integral is positive, a contradiction.

2. There is $s\le t$ such that $\ell_{s-}=L_t$, and $\ell_s<\ell_{s-}$. In this case there exists $t'>s$ such that $\ell_\theta\le\ell_{s-}$ for all $\theta\in[s,t']$. We first show

$$\int_{F^{-1}(\ell_{s-},\infty)} u(x,s)dx = 1. \tag{3.2}$$

Let $s_n\uparrow s$. Then $\ell_{s_n}\to\ell_{s-}$ and by assumption, $\ell_{s_n}\le\ell_{s-}$. Now,

$$\begin{aligned} \int_{F^{-1}(\ell_{s-},\infty)} u(x,s)dx &= \int_{F^{-1}(\ell_{s_n},\infty)} u(x,s_n)dx + \int_{F^{-1}(\ell_{s_n},\infty)} (u(x,s)-u(x,s_n))dx \\ &\quad - \int_{F^{-1}(\ell_{s_n},\ell_{s-})} u(x,s)dx. \end{aligned} \tag{3.3}$$

The first term on the right is 1, and the last term converges to zero as $n\to\infty$. As for the second term, since for all $\theta\in[s_n,s]$ one has $F^{-1}(\ell_{s-},\infty)\subset F^{-1}(\ell_\theta,\infty)$, one can integrate in (2.1)(i) and get

$$\int_{F^{-1}(\ell_{s-},\infty)} |u(x,s)-u(x,s_n)|dx \le \int_{s_n}^s \int_{\mathbb{R}^d} u(y,\theta)dyd\theta = s-s_n.$$

The above expression and the second term in (3.3) have the same limit by the assumed boundedness of u on $\mathbb{R}^d\times[0,s]$. This shows (3.2).

Using (3.2) and $F^{-1}(\ell_{s-},\infty)\subset F^{-1}(\ell_{t'},\infty)$, we have $\int_{F^{-1}(\ell_{t'},\infty)} u(x,s)dx=1$. We can thus repeat the argument above in 1, with $(s,\ell_{s-},t',\ell_{t'})$ in place of (s,ℓ_s,t,ℓ_t). Namely,

$$\begin{aligned} 1 = \int_{F^{-1}(\ell_{t'},\infty)} u(x,t')dx &= \int_{F^{-1}(\ell_{t'},\infty)} \Big[u(x,s) + \int_s^{t'} \int_{\mathbb{R}^d} u(y,\theta)\rho(y,x)dyd\theta\Big]dx \\ &\ge 1 + \int_s^{t'} \int_{F^{-1}(\ell_{s-},\infty)}^{\infty} \int_{F^{-1}(\ell_{s-},\infty)} u(y,\theta)\rho(y,x)dydxd\theta. \end{aligned}$$

The argument now completes exactly as in case 1. □

Lemma 3.2. *Let (u,ℓ) and (v,m) be solutions of (2.1). Then $(u,\ell)=(v,m)$.*

Proof. If $w\in L^1(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} w(x)dx=0$ then we have $\|w\|_1 = 2\|w^+\|_1 = 2\|w^-\|_1$, where we denote $\|\cdot\|_1 = \|\cdot\|_{L^1}$. If in addition $w\ge 0$ on some domain D then

$$2\|w^-\|_1 = 2\|w^- 1_{D^c}\|_1 \le 2\|w 1_{D^c}\|_1.$$

A similar statement holds with $w \le 0$ and w^+. Hence if w is either nonnegative on D or nonpositive on D,

$$\|w\|_1 \le 2\|w1_{D^c}\|_1. \tag{3.4}$$

Consider (x,t) such that $F(x) > \ell_t$. Then $F(x) > \ell_s$ for all $s \le t$. Therefore (2.1)(i) is valid with (x,t) replaced by (x,s) for all such s, and

$$u(x,t) = u_0(x) + \int_0^t \int_{\mathbb{R}^d} u(y,s)\rho(y,x)dyds.$$

Similarly, if $F(x) > m_t$, the above is satisfied by v. Denote $\Delta = u - v$. Consider (x,t) such that $F(x) > \ell_t \vee m_t$. Then

$$\Delta(x,t) = \int_0^t \int_{\mathbb{R}^d} \Delta(y,s)\rho(y,x)dyds. \tag{3.5}$$

Now, for each t, $\int_{\mathbb{R}^d} \Delta(x,t)dx = 1 - 1 = 0$. Moreover, in $F^{-1}(-\infty, \ell_t \vee m_t)$, either u or v vanishes, therefore $\Delta(\cdot,t)$ is either nonnegative or nonpositive. Hence we can apply (3.4) and then (3.5) to get

$$\begin{aligned}\int_{\mathbb{R}^d} |\Delta(x,t)|dx &\le 2\int_{F^{-1}(\ell_t \vee m_t, \infty)} |\Delta(x,t)|dx \\ &\le 2\int_{F^{-1}(\ell_t \vee m_t, \infty)} \int_0^t \int_{\mathbb{R}^d} |\Delta(y,s)|\rho(y,x)dydsdx \\ &\le 2\int_0^t \int_{\mathbb{R}^d} |\Delta(y,s)|dyds.\end{aligned}$$

The above is true for all $t \ge 0$, hence by Gronwall's lemma, the integral on the left vanishes for all t. This shows that for every t, $u(x,t) = v(x,t)$ for a.e. x. Next, for every t, both $u(\cdot,t)$ and $v(\cdot,t)$ are continuous in each of the domains $\{F(x) \le \ell_t \wedge m_t\}$, $\{\ell_t \wedge m_t < F(x) \le \ell_t \vee m_t\}$ and $\{F(x) > \ell_t \vee m_t\}$, and therefore must be equal everywhere. This shows $u = v$.

It remains to show that $\ell = m$. Since $u = v$, (ℓ, u) and (m,u) are solutions. Arguing by contradiction, assume that, say, $\ell_\theta < m_\theta$ for some θ. By right continuity, there exists an interval $[\theta, \theta_1]$ such that

$$\sup_{t\in[\theta,\theta_1]} \ell_t < \inf_{t\in[\theta,\theta_1]} m_t.$$

Let $\hat{\ell}$ be defined by $\hat{\ell}_t = m_t$ for $t < \theta_1$ and $\hat{\ell}_t = \ell_t$ for $t \ge \theta_1$. Then $\hat{\ell}$ is càdlàg . Moreover, the differential equation (2.1)(i) holds when $F(x) > \hat{\ell}_t$ (because it holds in the larger domain $F(x) > \ell_t$), and the vanishing condition (2.1)(ii) holds when $F(x) \le \hat{\ell}_t$ (because it holds in the larger domain $F(x) \le m_t$). Hence $(\hat{\ell}, u)$ is a solution. However, by construction, $\hat{\ell}$ is not a nondecreasing trajectory, which contradicts the monotonicity property proved earlier. This shows that $\ell = m$. □

3.2. Proof of Theorem 2.2

In this section it is proved first, in Lemma 3.3, that tightness holds, and then the main remaining task is to show that limits satisfy (2.1)(i). As outlines in Section 3.2, this is achieved by constructing upper and lower bounds on the density given in terms of limits of particle systems with piecewise constant boundary, for which convergence is a consequence of earlier work. These piecewise constant boundaries are constructed to form upper and lower envelopes of the prelimit free boundary ℓ^N.

Lemma 3.3. *The sequence of laws of $(\bar{\xi}^N, \ell^N)$ is C-tight. Moreover, every subsequential limit (ξ, ℓ) satisfies a.s.,*

$$\xi_t(F^{-1}(-\infty, \ell_t)) = 0, \text{ for all } t \in (0,\infty).$$

Proof. We first argue C-tightness of ℓ^N. Since J^N is a rate-N Poisson process, $\bar{J}^N = N^{-1}J^N$ converges in probability to the identity map from $\mathbb{R}_+$ to itself. Next, by construction, the path $t \mapsto \ell^N_t$ is nondecreasing, because when a new particle is born at t in the domain $F^{-1}[\ell^N_{t-}, \infty)$ one has $\ell^N_t \ge \ell^N_{t-}$, and if it is born outside this domain, it is removed immediately and $\ell^N_t = \ell^N_{t-}$. In particular, $\ell^N_t \ge \ell^N_0 \to \lambda_0$ in probability. Fix $T > 0$. We show that the random variables ℓ^N_T are tight. To this end, denote by ζ^N the configuration measure associated with non-local branching without removals. That is, they are constructed as our original system, but without removing any particles, and the systems are coupled so that at all times, the configuration of the original system is a subset of that of the enlarged one. Then there exists a (deterministic) finite measure on $\mathbb{R}^d$, ζ_T, such that $\bar{\zeta}^N_T \to \zeta_T$ in probability by [18, Theorem 5.3]. As a result, there exists a compact $K \subset \mathbb{R}^d$ such that $\limsup_N \mathbb{P}(\bar{\zeta}^N_T(K^c) > \frac{1}{2}) = 0$. Let $k = \max\{F(x) : x \in K\}$. Then $\limsup_N \mathbb{P}(\bar{\zeta}^N_T(F^{-1}(k,\infty)) > \frac{1}{2}) = 0$. Since ξ^N_T is dominated by ζ^N_T a.s.,

$$\mathbb{P}(\ell^N_T > k) = \mathbb{P}(\xi^N_T(F^{-1}(k,\infty)) = N) \le \mathbb{P}(\bar{\zeta}^N_T(F^{-1}(k,\infty)) \ge 1),$$

showing $\limsup_N \mathbb{P}(\ell^N_T > k) = 0$.

Next, for ℓ^N to increase over a time interval $[t, t+h]$ by more than δ, the particles located at time t in $D^N(t,\delta) := F^{-1}[\ell^N_t, \ell^N_t + \delta)$ must be removed by time $t+h$. Because ℓ^N is monotone, all particles in the initial configuration within the domain $D^N(t,\delta)$ are still present at time t. Hence the event $\ell^N_{t+h} > \ell^N_t + \delta$ is contained in

$$\xi^N_0(D^N(t,\delta)) \le J^N_{t+h} - J^N_t.$$

As a result, the event $w_T(\ell^N, h) > \delta$ is contained in

$$\inf_{t\in[0,T-h]} \bar{\xi}_0^N(D^N(t,\delta)) \le w_T(\bar{J}_t^N, h).$$

Hence for any $\delta, h, \varepsilon > 0$,

$$\mathbb{P}(w_T(\ell^N, h) > \delta) \le \mathbb{P}(\ell_T^N > k) + \mathbb{P}(w_T(\bar{J}^N, h) > \varepsilon) + \mathbb{P}\Big(\inf_{t\in[0,T-h]} \bar{\xi}_0^N(D^N(t,\delta)) \le \varepsilon,\ \ell_T^N \le k\Big).$$

Thus

$$\limsup_N \mathbb{P}(w_T(\ell^N, h) > \delta) \le \limsup_N \mathbb{P}(w_T(\bar{J}^N, h) > \varepsilon) + \limsup_N \mathbb{P}\Big(\inf_{a\in[\lambda_0,k]} \bar{\xi}_0^N(F^{-1}[a, a+\delta)) < \varepsilon\Big). \tag{3.6}$$

Let $\delta > 0$ be given. By assumption, $\eta(a,\delta) := \int_{F^{-1}(a,a+\delta)} u_0(x)dx > 0$, $a \ge \lambda_0$. Moreover, by the boundedness of u_0, $a \mapsto \eta(a,\delta)$ is continuous. Therefore $\inf_{a\in[\lambda_0,k]} \eta(a,\delta) > 0$. This and the convergence $\bar{\xi}_0^N \to \xi_0 = u_0(x)dx$ in probability show that one can find $\varepsilon > 0$ such that the second term on the r.h.s. of (3.6) vanishes. Given such ε, using the fact that limits of $\bar{J}^N$ are 1-Lipschitz, the first term on the r.h.s. of (3.6) also vanishes provided h is sufficiently small. This completes the proof of C-tightness of ℓ^N.

Next, C-tightness of $\bar{\xi}^N$ is shown. Denote by d_{L} the Levy–Prohorov metric on $\mathcal{P}(\mathbb{R}^d)$, which is compatible with weak convergence on this space. We will show that (i) for every $\varepsilon > 0$ and t there exists a compact set $K_{\varepsilon,t} \subset \mathcal{P}(\mathbb{R}^d)$ such that $\liminf_N \mathbb{P}(\bar{\xi}_t^N \in K_{\varepsilon,t}) > 1-\varepsilon$; and (ii) for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_N \mathbb{P}(w_T(\bar{\xi}^N, \delta) > \varepsilon) < \varepsilon.$$

This will establish C-tightness of $\bar{\xi}^N$ in view of [19, Corollary 3.7.4 (p. 129)] and because we use w rather than w' [19, (3.6.2) (p. 122)].

To show (i), let

$$K_n(r) = \{\gamma \in \mathcal{P}(\mathbb{R}^d) : \gamma(\mathbb{B}_r^c) < n^{-1}\}, \qquad n \in \mathbb{N},\ r \in (0,\infty).$$

By Prohorov's theorem, for any n_0 and sequence $\{r_n\}$, the closure of $K_{\ge n_0}(\{r_n\}) := \cap_{n\ge n_0} K_n(r_n)$, as a subset of $\mathcal{P}(\mathbb{R}^d)$, is compact. Suppose we show that there exists a sequence r_n such that, for every $t \in [0,T]$,

$$\liminf_N \mathbb{P}(\bar{\xi}_t^N \in K_n(r_n)) \ge 1 - 2^{-n}. \tag{3.7}$$

Then, given $\varepsilon > 0$, taking n_0 such that $\sum_{n\ge n_0} 2^{-n} < \varepsilon$, it would follow that

$$\liminf_N \mathbb{P}(\bar{\xi}_t^N \in \overline{K_{\ge n_0}(\{r_n\})}) > 1 - \varepsilon,$$

showing that (i) holds. To this end, note that the convergence of $\bar{\zeta}_T^N$ implies that there exists a sequence $\{r_n\}$ such that

$$\liminf_N \mathbb{P}(\bar{\zeta}_T^N(\mathbb{B}_{r_n}^c) < n^{-1}) > 1 - 2^{-n}.$$

But

$$\sup_{t\in[0,T]} \bar{\xi}_t^N(\mathbb{B}_{r_n}^c) \le \bar{\zeta}_T^N(\mathbb{B}_{r_n}^c),$$

hence (3.7) follows, and (i) is proved.

Next we show (ii). Given a set $C \subset \mathbb{R}$ let C^ϵ denote its ε-neighborhood. Let $0 \le s < t \le T$ be such that $t - s \le \delta$. Then for any Borel set C, one has

$$|\bar{\xi}_t^N(C) - \bar{\xi}_s^N(C)| \le J_t^N - J_s^N.$$

Hence, on the event $w_T(\bar{J}^N, \delta) \le \varepsilon$,

$$\bar{\xi}_s^N(C) \le \bar{\xi}_t^N(C^\epsilon) + \varepsilon, \qquad \text{and} \qquad \bar{\xi}_t^N(C) \le \bar{\xi}_s^N(C^\epsilon) + \varepsilon,$$

and thus $d_{\mathrm{L}}(\bar{\xi}_s^N, \bar{\xi}_t^N) \le \varepsilon$. This shows that for sufficiently small δ,

$$\limsup_N \mathbb{P}(w_T(\bar{\xi}^N, \delta) > \varepsilon) \le \varepsilon,$$

and the proof of (ii) is complete.

For the second assertion of the lemma, note that by the definition of ℓ^N, one has for all N, $\bar{\xi}_t^N(F^{-1}(-\infty, \ell_t^N)) = 0$. Invoking Skorohod's representation, one has $(\bar{\xi}^N, \ell^N) \to (\xi, \ell)$ a.s. along the convergent subsequence. Hence for $t > 0$ and $\varepsilon > 0$, because $F^{-1}(-\infty, \ell_t - \varepsilon)$ is open,

$$\xi_t(F^{-1}(-\infty, \ell_t - \varepsilon)) \le \liminf_N \bar{\xi}_t^N(F^{-1}(-\infty, \ell_t - \varepsilon))$$

$$\leq \liminf_N \bar{\xi}^N_t(F^{-1}(-\infty, \ell^N)) = 0,$$

a.s. Taking $\varepsilon \downarrow 0$, $\xi_t(F^{-1}(-\infty, \ell_t)) = 0$, a.s. To deduce the a.s. statement simultaneously for all t, it suffices to note that for $t_n \downarrow t$, by monotonicity of ℓ, one has

$$\xi_t(F^{-1}(-\infty, \ell_t)) \leq \liminf_n \xi_{t_n}(F^{-1}(-\infty, \ell_t)) \leq \liminf_n \xi_{t_n}(F^{-1}(-\infty, \ell_{t_n})) = 0. \quad \square$$

Proof of Theorem 2.2. In view of Lemmas 3.2 and 3.3, the proof will be complete once it is shown that for every limit (ξ, ℓ) there exists a measurable density u such $(u, \ell) \in \mathcal{X}$ and (u, ℓ) satisfies (2.1). Fix a convergent subsequence and denote its limit by (ξ, ℓ).

To argue the existence of a density let us go back to the particle system with no removals, mentioned in the proof of Lemma 3.3. The normalized process $\bar{\zeta}^N \to \zeta$, in probability, where ζ is deterministic and for every t, ζ_t has a density, as follows from [18, Theorem 5.3 and Proposition 5.4]. Throughout what follows, denote this density by $z(\cdot, t)$.

For any bounded continuous $g : \mathbb{R}^d \to [0, \infty)$, one has $\mathbb{E}\int g(x)\bar{\xi}^N_t(dx) \leq \mathbb{E}\int g(x)\bar{\zeta}^N_T(dx)$, $t \in [0, T]$. Hence $\mathbb{E}\int g(x)\xi_t(dx) \leq \int g(x)\zeta_T(dx)$, $t \in [0, T]$. In particular, $\xi_t(dx) \ll dx$, $t \leq T$, and since T is arbitrary, this statement holds for all t. Assuming $\xi = 0$ outside the full measure event, we finally obtain that for every $(t, \omega) \in (0, \infty) \times \Omega$, $\xi_t(dx, \omega) \ll dx$. We can now appeal to [20, Theorem 58 in Chapter V (p. 52)] and the remark that follows. The measurable spaces denoted in [20] by $(\Omega, \mathcal{F})$ and $(T, \mathcal{T})$ are taken to be $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and $((0, \infty) \times \Omega, \mathcal{B}((0, \infty)) \otimes \mathcal{F})$, respectively. According to this result there exists a $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}((0, \infty)) \otimes \mathcal{F}$-measurable function $u(x, t, \omega)$, such that for every $(t, \omega) \in (0, \infty) \times \Omega$, $u(\cdot, t, \omega)$ is a density of $\xi_t(dx, \omega)$ with respect to dx. We also have $u(x, t, \omega) \leq z(x, t)$.

Items (ii), (iii) and (iv) of (2.1) can be verified plainly: In view of Lemma 3.3, u has a version satisfying $u(x, t) = 0$ for all (x, t) such that $F(x) < \ell_t$, and this extends to $F(x) \leq \ell_t$ using Assumption 2.1(i) by which Leb $F^{-1}\{\ell_t\} = 0$ for all t. This verifies (2.1)(ii). Items (iii) and (iv) are obvious from our assumptions. It remains to show (2.1)(i), which by the nondecreasing property of ℓ can be expressed as

$$u(x, t) = u_0(x) + \int_0^t \int_{\mathbb{R}^d} u(y, s)\rho(y, x)dyds, \qquad (x, t) : F(x) > \ell_t, t > 0. \tag{3.8}$$

Note that as soon as the above is established, the uniform continuity of ρ and the local integrability of u imply the existence of a version of u for which $x \mapsto u(x, t)$ and $t \mapsto \partial_t u(x, t)$ are continuous, hence $(u, \ell) \in \mathcal{X}$. We will achieve (3.8) in several steps.

Step 1. *Integro-differential systems with piecewise constant boundary.* Fix $T > 0$ throughout. For $\delta, \varepsilon \in (0, 1)$ such that $H := \delta^{-1}T \in \mathbb{N}$, let $M = M(\delta, \varepsilon)$ denote the collection of right-continuous piecewise constant nondecreasing trajectories $m : [0, T) \to \mathbb{R}$ such that, with $t_j = j\delta$, for each $j = 0, 1, \ldots, H-1$, on $[t_j, t_{j+1})$, m takes a constant value in $\varepsilon\mathbb{Z}$. Denote $a_j(m) = m(t_j)$. For $m \in M$ and $a_j = a_j(m)$, consider the set of equations

$$\begin{cases} \partial_t u(x, t) = \int_{\mathbb{R}^d} u(y, t)\rho(y, x)dy & (x, t) : F(x) > a_j,\ t \in (t_j, t_{j+1}), \\ u(x, t) = 0 & (x, t) : F(x) \leq a_j,\ t \in (t_j, t_{j+1}), \\ u(x, t_j) = u(x, t_j-)1_{\{F^{-1}[a_j, \infty)\}}(x) & 1 \leq j \leq H-1, \\ u(x, 0) = u_0(x)1_{\{F^{-1}[a_0, \infty)\}}(x). \end{cases} \tag{3.9}$$

It is clear, by induction, that this set uniquely determines a solution, denoted $u^{(m)}$. If $\zeta^{(m)}_t(dx) = u^{(m)}(x, t)dx$, $t \in [0, T)$, then the map $M \ni m \mapsto \zeta^{(m)} \in D([0, T), \mathcal{M}_+(\mathbb{R}^d))$ is denoted by $\mathcal{Z}$.

Step 2. *A continuity property of $\mathcal{Z}$.* We prove the following claim. There exists a function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\gamma(0+) = 0$ and the following holds. Let $k, m \in M$ and $w = u^{(k)}$, $v = u^{(m)}$. Denote $\Delta = w - v$. Then

$$\sup_{t<T} \|\Delta(\cdot, t)\|_1 \leq \gamma(\|m - k\|^*_T), \qquad k, m \in M. \tag{3.10}$$

To prove the claim, denote

$$\eta(k, m) = \max_{j \leq H-1} \zeta_T(F^{-1}(a_j(k) \wedge a_j(m), a_j(k) \vee a_j(m))).$$

Note that

$$w(x, t) = w_0(x) + \int_0^t \int_{\mathbb{R}^d} w(y, s)\rho(y, x)dyds, \qquad (x, t) : F(x) > k_t, t > 0, \tag{3.11}$$

$$v(x, t) = v_0(x) + \int_0^t \int_{\mathbb{R}^d} v(y, s)\rho(y, x)dyds, \qquad (x, t) : F(x) > m_t, t > 0, \tag{3.12}$$

and w (resp., v) vanishes to the left of k (resp., m). Also note that $w(x, t) \vee v(x, t) \leq z(x, T)$. For $t \in [0, T)$,

$$\begin{aligned} \|\Delta(\cdot, t)\|_1 &\leq \|\Delta(\cdot, 0)\|_1 + \int_{F^{-1}(k_t, \infty)\triangle F^{-1}(m_t, \infty)} (w \vee v)(x, t)dx \\ &\quad + \int_{F^{-1}(k_t \vee m_t, \infty)} \int_0^t \int_{\mathbb{R}^d} |\Delta(y, s)|\rho(y, x)dyds dx \\ &\leq \|\Delta(\cdot, 0)\|_1 + \eta(k, m) + \tilde{c}\int_0^t \|\Delta(\cdot, s)\|_1 ds. \end{aligned}$$

Since $\|\Delta(\cdot,0)\|_1 \le \eta(k,m)$, Gronwall's lemma gives

$$\|\Delta(\cdot,t)\|_1 \le 2\eta(k,m)e^{\bar{c}T}, \qquad t<T.$$

Hence (3.10) will be proved once it is shown that $U(0+)=0$, where

$$U(\kappa)=\sup\{\zeta_T(F^{-1}(b,b+\kappa)) : b\in\mathbb{R}\}, \qquad \kappa>0.$$

Assuming the contrary, there exists $\kappa'>0$ and $\{b_n\}$ such that, along a subsequence,

$$\zeta_T(F^{-1}(b_n,b_n+n^{-1}))>\kappa'. \tag{3.13}$$

Now, $\{b_n\}$ must be bounded. For if there is a subsequence with $b_n\to\infty$, by the continuity of F, for every compact K the set $F^{-1}(b_n,\infty)\cap K$ must be empty for large n, which shows that (3.13) cannot hold. If there is a subsequence $b_n\to-\infty$ then $F^{-1}(-\infty,b_n+n^{-1})\cap K$ must be empty for all large n and again (3.13) cannot hold. Hence b_n is bounded. Let b be a limit. Then, given $\kappa_1>0$, $(b_n,b_n+n^{-1})\subset(b-\kappa_1,b+\kappa_1)$ for large n along a subsequence, showing

$$\kappa'\le\zeta_T(F^{-1}(b-\kappa_1,b+\kappa_1)).$$

Hence $\zeta_T(F^{-1}\{b\})>0$, contradicting our assumption $\mathrm{Leb}(F^{-1}\{a\})=0$ for every a. This proves (3.10).

Step 3. *Particle systems with piecewise constant boundary*. Given $m\in M$ we construct a particle system in which, for every $t\in[0,T)$, all particles lie within $F^{-1}[m_t,\infty)$. Its initial configuration is given by the restriction of $\{x_i\}$ to $F^{-1}[m_0,\infty)$. During any interval (t_j,t_{j+1}), the reproduction is according to ρ, but every newborn outside of $F^{-1}[a_j(m),\infty)$ is immediately removed. If for $j\in\{1,\dots,H-1\}$, t_j is a continuity point of m, nothing happens at this time. Otherwise, m necessarily performs a positive jump, hence the domain $F^{-1}[m,\infty)$ decreases. At this time, all particles outside $F^{-1}[a_j(m),\infty)$ are removed. Denote the configuration process by $\zeta^{(m),N}$.

We show that for every $m\in M$, $\bar{\zeta}^{(m),N}\to\zeta^{(m)}:=\mathcal{Z}(m)$ in probability, uniformly on $[0,T)$. For the initial condition, we have $\bar{\zeta}_0^{(m),N}=\bar{\xi}_0^N 1_{\{F^{-1}[a_0,\infty)\}}$. The boundary of the domain $F^{-1}[a_0,\infty)$ has Lebesgue measure zero, as follows from Assumption 2.1(i) upon noting that $\partial F^{-1}[a,\infty)\subset F^{-1}\{a\}$ for every $a\in\mathbb{R}$. Because the limit ξ_0 has a density, this implies $\bar{\zeta}_0^{(m),N}\to\xi_0 1_{\{F^{-1}[a_0,\infty)\}}$ in probability.

Assume that for $j\ge0$, $\bar{\zeta}_{t_j}^{(m),N}\to\zeta_{t_j}^{(m)}$. Then the convergence on the interval (t_j,t_{j+1}) to a measure-valued trajectory which has a density satisfying the integro-differential equation in (3.9) is as in the proof of [1, Proposition 2.1]; the proof there, based on [18], is for $d=1$ and the domain $\mathbb{R}$, but the same proof is valid for $d\ge1$ an arbitrary domain of $\mathbb{R}^d$, because that is the generality of the results of [18]. Because the trajectory is continuous on (t_j,t_{j+1}), the convergence is uniform there. Next, at the trimming time t_{j+1}, the convergence follows by the same argument as for $t=0$.

Step 4. *Relation to the original particle system*. Consider now, for each N, two stochastic processes, k^N and m^N, with sample paths in M, defined as follows. For $x\in\mathbb{R}$, let $P_\varepsilon^-(x)=\max\{y\in\varepsilon\mathbb{Z} : y\le x\}$, $P_\varepsilon^+(x)=\min\{y\in\varepsilon\mathbb{Z} : y\ge x\}$, and let

$$k_t^N=P_\varepsilon^-\Big(\inf_{s\in[t_j,t_{j+1})}\ell_s^N\Big)-\varepsilon, \qquad m_t^N=P_\varepsilon^+\Big(\sup_{s\in[t_j,t_{j+1})}\ell_s^N\Big)+\varepsilon, \qquad t\in[t_j,t_{j+1}).$$

By construction,

$$k_t^N<\ell_t^N<m_t^N, \qquad t\in[0,T). \tag{3.14}$$

Constructing a particle system with piecewise constant boundary, as in Step 2, makes perfect sense even when the trajectories are random members of M that anticipate the future of ξ^N, such as k^N and m^N. To construct these systems, one first evaluates ℓ^N, and based on it, k^N and m^N, and then uses these random trajectories to construct the particle systems as in Step 2. A key point is that, thanks to (3.14), one can couple the three systems in such a way that the two auxiliary configuration measures form lower and upper bounds on ξ^N in the sense of measure inequality. In particular, we have the following.

Lemma 3.4. *There exists a coupling such that, a.s.,*

$$\zeta_t^{(m^N),N}\sqsubset\xi_t^N\sqsubset\zeta_t^{(k^N),N}, \qquad t\in[0,T), \tag{3.15}$$

where the two particle configurations corresponding to the piecewise constant boundaries k^N and m^N are denoted by $\zeta^{(k^N),N}$ and $\zeta^{(m^N),N}$.

The standard but tedious proof of this lemma appears in the Appendix.

Since the maps $P_\varepsilon^\pm$ are not continuous, k^N and m^N may not converge in law. However, the tightness of the laws of ℓ^N and the specific structure of M clearly give tightness of (k^N,m^N). Consider then any convergent subsequence $(\bar{\xi}^N,\ell^N,k^N,m^N)\Rightarrow(\xi,\ell,\hat{k},\hat{m})$. We claim that the limit in law of $(\bar{\xi}^N,\ell^N,k^N,m^N,\bar{\zeta}^{(k^N),N},\bar{\zeta}^{(m^N),N})$ exists and is given by

$$(\xi,\ell,\hat{k},\hat{m},\mathcal{Z}(\hat{k}),\mathcal{Z}(\hat{m})).$$

To simplify the notation, we prove the claim only for the 3-tuple $(\bar{\xi}^N,m^N,\bar{\zeta}^{(m^N),N})$; the proof for the 6-tuple is very similar. Because the claim is concerned with convergence of probability measures, it suffices to prove vague convergence. Let $f : D_T\to\mathbb{R}$ be compactly supported, where $D_T:=D([0,T],\mathcal{M}(\mathbb{R}^d))\times D([0,T],\mathbb{R})\times D([0,T],\mathcal{M}(\mathbb{R}^d))$. Let M_f be the set of $m\in M$ for which $f(\cdot,m,\cdot)\not\equiv0$ and note that $\#M_f<\infty$ because for $c>0$ there are only finitely many $m\in M$ with $\|m\|_T^*<c$. By the uniform continuity of f, one has $\gamma_f(0+)=0$, where

$$\gamma_f(\kappa)=\sup\{|f(\alpha,m,\beta)-f(\alpha,m,\beta')| : (\alpha,m,\beta)\in D_T,\ d_{\mathrm{L}}(\beta,\beta')<\kappa\}.$$

Now,

$$\mathbb{E}f(\bar{\xi}^N, m^N, \bar{\zeta}^{(m^N),N}) = \sum_{m\in M_f} \mathbb{E}[1_{\{m\}}(m^N) f(\bar{\xi}^N, m, \bar{\zeta}^{(m),N})] = \sum_{m\in M_f} \mathbb{E}[1_{\{m\}}(m^N) f(\bar{\xi}^N, m, \mathcal{Z}(m))] + W^N,$$

where

$$|W^N| \le \sum_{m\in M_f} \mathbb{E}[\gamma_f(d_{\mathrm{L}})(\bar{\zeta}^{(m),N}, \mathcal{Z}(m))] \to 0,$$

by the convergence in probability proved in Step 3. Because each $m \in M$ is isolated from all other members of M, the convergence $(\bar{\xi}^N, m^N) \Rightarrow (\xi, \hat{m})$ implies $\mathbb{E}[1_{\{m\}}(m^N) g(\bar{\xi}^N)] \to \mathbb{E}[1_{\{m\}}(\hat{m}) g(\xi)]$ whenever g is bounded and continuous. This gives

$$\mathbb{E}f(\bar{\xi}^N, m^N, \bar{\zeta}^{(m^N),N}) \to \sum_{m\in M_f} \mathbb{E}[1_{\{m\}}(\hat{m}) f(\xi, m, \mathcal{Z}(m))] = \mathbb{E}[f(\xi, \hat{m}, \mathcal{Z}(\hat{m}))],$$

proving the claim.

As a consequence of the above convergence and (3.15), we obtain a bound on the density u, namely, for every t and a.e. x,

$$\hat{v}(x,t) \le u(x,t) \le \hat{w}(x,t), \tag{3.16}$$

where $\hat{v} = u^{(\hat{m})}$ and $\hat{w} = u^{(\hat{k})}$ are the random densities corresponding to $\mathcal{Z}(\hat{m})$ and $\mathcal{Z}(\hat{k})$.

Step 5. *Completing the proof.* Denoting $\tilde{\Delta} = u - \hat{v}$, $\hat{\Delta} = \hat{w} - \hat{v}$, and $\hat{q} = \|\hat{m} - \hat{k}\|_T^*$, we have by (3.16), $\|\tilde{\Delta}(\cdot,t)\|_1 \le \|\hat{\Delta}(\cdot,t)\|_1 \le \gamma(\hat{q})$, where (3.10) is used.

Note that $\hat{w}$ and $\hat{v}$ satisfy (3.11)–(3.12), with k and m replaced by the random $\hat{k}$ and $\hat{m}$. Let (x,t) be such that $F(x) > \hat{m}_t$, by which $F(x) > \ell_t \ge \hat{k}_t$. Then

$$u(x,t) = \hat{v}(x,t) + \tilde{\Delta}(x,t) = u_0(x) - \tilde{\Delta}_0(x) + \int_0^t \int_{\mathbb{R}^d} u(y,s)\rho(y,x)dyds - \int_0^t \int_{\mathbb{R}^d} \tilde{\Delta}(y,s)\rho(y,x)dyds + \tilde{\Delta}(x,t).$$

Denoting, for $(x,t) \in \mathbb{R}^d \times [0,T)$,

$$\Psi(x,t) = \left| u(x,t) - u_0(x) - \int_0^t \int_{\mathbb{R}^d} u(y,s)\rho(y,x)dyds \right|,$$

we therefore have

$$\int_{\mathbb{R}^d} \Psi(x,t) 1_{\{F(x)>\hat{m}_t\}} dx \le (2 + \tilde{c}T)\gamma(\hat{q}).$$

Moreover,

$$\Psi(x,t) \le 2z(x,T) + T\int_{\mathbb{R}^d} z(y,T)\tilde{c}\tilde{\rho}(x-y)dyds.$$

Analogously to U, define $\tilde{U}(\kappa) = \sup\{\int_{F^{-1}(b,b+\kappa)} \tilde{\rho}(x)dx : b \in \mathbb{R}\}$, and note, by similar reasoning, that $\tilde{U}(0+) = 0$. Then

$$\int_{\mathbb{R}^d} \Psi(x,t) 1_{\{\ell_t < F(x) \le \hat{m}_t\}} dx \le 2U(\hat{q}) + T\tilde{c}\,|\zeta_T|\,\tilde{U}(\hat{q}).$$

Hence there exists a constant c such that

$$\psi := \sup_{t<T} \int_{\mathbb{R}^d} \Psi(x,t) 1_{\{F(x)>\ell_t\}} dx \le c(\gamma(\hat{q}) + U(\hat{q}) + \tilde{U}(\hat{q})).$$

Given $\varepsilon, \delta' > 0$ choose $\delta > 0$ so small that $\mathbb{P}(\Omega_{\delta,\varepsilon}) > 1 - \delta'$, where $\Omega_{\delta,\varepsilon} = \{w_T(\ell,\delta) < \varepsilon\}$. By the definition of m^N one has for every j, $m^N_{t_j} \le \sup_{[t_j,t_{j+1})} \ell^N + 2\varepsilon$, hence $\hat{m}_{t_j} \le \sup_{[t_j,t_{j+1})} \ell + 2\varepsilon$. An analogous statement holds for $\hat{k}$. This gives, on $\Omega_{\delta,\varepsilon}$, $\hat{m}_t - \hat{k}_t \le 5\varepsilon$ for all $t < T$, hence $\hat{q} \le 5\varepsilon$. As a result,

$$\mathbb{P}(\psi > c(\gamma(5\varepsilon) + U(5\varepsilon) + \tilde{U}(5\varepsilon))) < \delta'.$$

Finally, $\delta' \to 0$ followed by $\varepsilon \to 0$ shows $\psi = 0$ a.s. This shows (3.8) and completes the proof. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. Proof of Lemma 3.4

The three particle systems will be constructed out of a single branching random walk (with labeled particles), which is a non-local branching model without selection. In this system, the initial N particles are located at x^i, $1 \le i \le N$. Each particle has a rate-1 Poisson clock according to which it gives birth at location drawn according to $\rho(y,\cdot)$, y being the particle's location. The first N particles are labeled $1,\dots,N$, and for $i \ge N+1$, the ith particle to be born (regarding the initial N particles as being born at 0) is labeled i. Let the birth time of particle $i \ge 1$ be denoted by σ_i. Let $\tau_{i,j}$, $j \ge 1$, denote the times particle i gives birth, and let $\tau_i(t) = \min\{\tau_{i,j} : \tau_{i,j} > t\}$ denote the next time after t when particle i gives birth. Let X_i denote the position of particle i. With this notation, the configuration measure of the branching random walk is

$$\zeta_t^N = \sum_{i \in \mathcal{Q}(t)} \delta_{X_i}, \qquad \mathcal{Q}(t) = \{i : t \ge \sigma_i\}.$$

For a nonempty finite set $S \subset \mathbb{N}$, denote by $\arg\min S$ the particle $i \in S$ for which one has $F(X_i) \le F(X_j)$ for all $j \in S$, with ties broken according to the particle labels.

The system studied in this paper can be represented as

$$\xi_t^N = \sum_{i \in \mathcal{R}(t)} \delta_{X_i},$$

where $\mathcal{R}(t) \subset \mathcal{Q}(t)$ is defined inductively as follows. Starting with time 0, let $\mathcal{R}(0) = \{1,\dots,N\}$ and $\theta_0 = 0$. Next, for $k \ge 0$, given θ_k and $\{\mathcal{R}(t), t \le \theta_k\}$, define θ_{k+1} and $\{\mathcal{R}(t), t \le \theta_{k+1}\}$ in the following way.

Define $\theta_{k+1} = \min\{\tau_i(\theta_k) : i \in \mathcal{R}(\theta_k)\}$ and let i_{k+1} be the particle born at θ_{k+1}. Thus, up to θ_{k+1} no particle among $\mathcal{R}(\theta_k)$ gives birth, whereas at θ_{k+1}, a particle in this set gives birth to particle i_{k+1}. Let $\mathcal{R}(t) = \mathcal{R}(\theta_k)$ for $t \in (\theta_k, \theta_{k+1})$ and

$$\mathcal{R}(\theta_{k+1}) = \mathcal{R}(\theta_k) \cup \{i_{k+1}\} - \arg\min(\mathcal{R}(\theta_k) \cup \{i_{k+1}\}).$$

It is clear that $\theta_k \to \infty$ a.s. as $k \to \infty$, and therefore $\mathcal{R}(t)$ is defined for all t, a.s., completing the construction of ξ_t^N.

Recall that the set M consists of piecewise constant trajectories whose jump times are denoted by t_j. Let $T(t) = \min\{t_j : t_j > t\}$. For an arbitrary $m \in M$ (where m will later stand for either m^N or k^N), let

$$\zeta_t^{(m),N} = \sum_{i \in \mathcal{S}(m;t)} \delta_{X_i}.$$

Here, $\mathcal{S}(m;t)$, abbreviated to $\mathcal{S}(t)$, is constructed as follows. Initially, $\mathcal{S}(0) = \{1,\dots,N\}$, and $s_k = 0$. Given s_k and the construction of $\mathcal{S}(t)$, $t \le s_k$, let

$$s_{k+1} = \min\{\tau_i(s_k) : i \in \mathcal{S}(s_k)\} \wedge T(s_k).$$

Thus at s_{k+1} either (a) one of the particles in $\mathcal{S}(s_k)$ gives birth, or (b) one of the t_j has been reached. In case (a), denote the newborn particle by i_{k+1} and let

$$\mathcal{S}(s_{k+1}) = \begin{cases} \mathcal{S}(s_k) \cup \{i_{k+1}\} & \text{if } F(X_{i_{k+1}}) \ge m(s_k), \\ \mathcal{S}(s_k) & \text{if } F(X_{i_{k+1}}) < m(s_k). \end{cases}$$

Thus the new particle is accepted if its fitness is $\ge m(s_k) = m(s_{k+1})$. In case (b),

$$\mathcal{S}(s_{k+1}) = \mathcal{S}(s_k) - \{i \in \mathbb{N} : F(X_i) < m(s_{k+1})\},$$

rejecting all particles whose fitness is $< m(s_{k+1})$. This completes the construction of $\zeta^{(m),N}$.

In view of the inequalities (3.14), the lemma will be proved once it is shown that

$$\text{if } m(t) < \ell^N(t) \text{ for all } t \text{ then } \mathcal{R}(t) \subset \mathcal{S}(t) \text{ for all } t, \tag{A.1}$$

$$\text{if } m(t) > \ell^N(t) \text{ for all } t \text{ then } \mathcal{S}(t) \subset \mathcal{R}(t) \text{ for all } t. \tag{A.2}$$

We only provide the proof of (A.1) as that of (A.2) is similar. Arguing by contradiction, assume (A.1) is false, and let s be the first time it is violated. That is, at s, for the first time, there is a particle in $\mathcal{R}(s)$ that is not in $\mathcal{S}(s)$. At such s, one of these must occur:

- A new particle whose fitness is $\ge \ell^N(s)$ has been introduced into $\mathcal{R}(s)$ (and not immediately rejected), that is a child of a particle in the set $\mathcal{R}(s-)$. Because $\mathcal{R}(s-) \subset \mathcal{S}(s-)$, this particle was also introduced into $\mathcal{S}(s)$, and because $m(s) < \ell^N(s)$, it was not immediately rejected. This contradicts the assumption $\mathcal{R}(s) \not\subset \mathcal{S}(s)$.

- At s, m makes a jump upward, that is, $m(s) > m(s-)$. As a result, some particles in $\mathcal{S}(s-)$ are removed and do not appear in $\mathcal{S}(s)$, whereas at least one of them appears in $\mathcal{R}(s)$. Thus the fitness of this particle is $\ge \ell^N(s)$, which contradicts the fact that it also must be $< m(s)$.

This completes the proof of Lemma 3.4. □

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