Trimmed Branching Brownian Motion with Discontinuous Drift

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Abstract

We consider a system of branching Brownian particles on the real line with bounded, possibly discontinuous drift b, and with a selection rule that removes one particle from the most crowded cells of a uniform partition of the line at each branching event so that the number of particles is preserved. As the number of particles tends to infinity, with the cell size tending to zero (sufficiently slowly), we prove convergence in probability of the empirical measure process to a deterministic limit characterized by the parabolic free obstacle problem

$$\partial_t u = \tfrac{1}{2}\partial_{xx} u - \partial_x(bu) + u - \beta.$$

Here the density u has unit mass. Disintegrating the removal measure β as $\beta_t(\mathrm{d}x)\mathrm{d}t$, the probability measure β_t is supported on the spatial maximizers of $u(t,\cdot)$ for a.e. t. We establish existence and uniqueness of weak solutions in a suitable class of continuous densities, extending earlier results about this PDE to discontinuous drifts. We then consider an inverse problem: Given a target removal measure, find a drift for which the PDE admits a stationary solution with that removal measure. We show that this problem is solvable for every compactly supported Borel probability measure.

1 Introduction and Main Results

We study a version of n-particle branching Brownian motion with drift in which each branching event is accompanied by the removal of a particle from a most populated spatial cell. This mechanism trims the population while keeping its size fixed, as in the branching random walk model of [AMW26]. We consider the particle system together with its macroscopic density evolution, described by a parabolic free obstacle problem. Mass is removed from the top of the density profile, whose height serves as an upper obstacle. This height is not prescribed but is determined together with the density under the unit mass constraint; hence the term 'free'.

The model can be interpreted as describing a population of mobile organisms in a habitat with limited resources. As in earlier models of branching and selection [BDMM07], this constraint motivates the assumption of a fixed population size. In our model, each birth is balanced by a death. Individuals reproduce at their current location and move during their lifetimes: Diffusion describes random dispersal, drift represents directed movement or transport by the environment. Models that combine movement, reproduction and competition have been studied in population dynamics [HL04]. Our removal rule represents competition for local resources: Individuals die in locations where crowding is greatest. Mortality also depends on local crowding in the model of [FM04], where individuals are motionless and dispersal occurs at birth.

Examples in [AMW26] exhibit both absolutely continuous and atomic removal measures. Some involve discontinuous drifts and lie outside the scope of the well-posedness and convergence results established there. They motivate two questions that we pursue here: whether the theory extends to less regular drifts, and how rich a class of removal measures can be obtained by choosing different drifts.

The Particle System

We consider n particles on the real line. Between branching and selection events, the particles evolve independently as Brownian motions with drift b. Each particle branches at rate one, producing an offspring at its current position. The exponential branching clocks of rate 1 are independent of one another and of the Brownian motions. Space is partitioned into cells of length ε. Each branching event is accompanied by the removal of a particle chosen uniformly from those occupying the most populated cells, so that the population size remains equal to n. The removal choice is based on the configuration immediately before branching and, conditional on this configuration, is independent of the choice of the branching particle.

Equivalently, we may transfer the particle selected for removal to the position of the branching particle. Giving the offspring the label of the removed particle allows us to describe the system using a fixed set of n labels. After

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each event, the diffusion motion resumes from the updated positions, using the same Brownian motions for the fixed labels, and the mechanism repeats.

We now give a precise construction. Fix $n \in \mathbb{N}$ and $\varepsilon > 0$, and write $[n] = \{1, \dots, n\}$. Let μ_0 be a probability measure on $\mathbb{R}$ with density $u_0 \in \mathcal{C}_0(\mathbb{R}, \mathbb{R})$. Here $\mathcal{C}_0(\mathbb{R}, \mathbb{R})$ denotes the space of continuous functions on $\mathbb{R}$ that vanish at infinity. We assume that $b : \mathbb{R} \to \mathbb{R}$ is bounded and Borel measurable. Define the cells

$$C_k^\varepsilon = [\varepsilon k, \varepsilon(k+1)), \qquad k \in \mathbb{Z}.$$

Now consider a configuration $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Let

$$\kappa_k^\varepsilon(x) = |\{i \in [n] : x_i \in C_k^\varepsilon\}|$$

be the number of particles in cell k, and let

$$A^\varepsilon(x) = \mathrm{argmax}_{k \in \mathbb{Z}} \kappa_k^\varepsilon(x) = \{k \in \mathbb{Z} : \kappa_k^\varepsilon(x) \geq \kappa_{k'}^\varepsilon(x) \text{ for all } k' \in \mathbb{Z}\}$$

be the set of indices of the most populated cells. For any $k \in A^\varepsilon(x)$, the total number of particles eligible for removal is $\kappa_k^\varepsilon(x)|A^\varepsilon(x)|$ so that uniform selection among these particles assigns each of them removal probability $1/(\kappa_k^\varepsilon(x)|A^\varepsilon(x)|)$.

To describe the branching and selection events, we introduce a Poisson random measure π^n with intensity measure

$$\hat{\pi}^n = \lambda_{(0,\infty)} \otimes \zeta_{[n]} \otimes \zeta_{[n]} \otimes \zeta_{\mathbb{Z}} \otimes \lambda_{[0,1]}$$

on $(0, \infty) \times [n] \times [n] \times \mathbb{Z} \times [0, 1]$. Here λ_D and ζ_D denote Lebesgue measure and counting measure restricted to D, respectively. The marks i and j will label the particle selected for removal and the branching particle, respectively; k will be the cell containing particle i; and θ will be an auxiliary mark used to implement the removal probabilities. We set

$$g^{n,\varepsilon}(x; i, k, \theta) = \mathbb{1}_{\{\theta \kappa_k^\varepsilon(x)|A^\varepsilon(x)| \leq 1,\, x_i \in C_k^\varepsilon,\, k \in A^\varepsilon(x)\}},$$

for $x \in \mathbb{R}^n$, $i \in [n]$, $k \in \mathbb{Z}$ and $\theta \in [0, 1]$. For a fixed configuration x, integrating $g^{n,\varepsilon}$ over θ gives rate $1/(\kappa_k^\varepsilon(x)|A^\varepsilon(x)|)$ for each candidate pair (i, j) with $x_i \in C_k^\varepsilon$ and $k \in A^\varepsilon(x)$. Summing over the eligible removal labels and their cells gives total rate 1 for each branching label j. Thus the removal label is uniform over the eligible particles, and the Poisson construction realizes the mechanism described above.

We now define the dynamics of the particles. Let $X^n(0) \sim \mu_0^{\otimes n}$ be the initial configuration, independent of π_n, and let $W^n = (W_i^n)_{i \in [n]}$ be a family of n independent standard Brownian motions, independent of $X^n(0)$ and π_n. Denoting by $X_i^{n,\varepsilon}(t)$ the position of particle $i \in [n]$ at time $t \geq 0$ and writing $X^{n,\varepsilon}(t) = (X_i^{n,\varepsilon}(t))_{i \in [n]}$, we define the stochastic process $X^{n,\varepsilon;b,u_0} = X^{n,\varepsilon}$ by

$$\begin{aligned} X^{n,\varepsilon}(t) = X^n(0) + W^n(t) + \int_0^t (b(X_i^{n,\varepsilon}(s)))_{i \in [n]} \mathrm{d}s \\ + \int_{(0,t] \times [n] \times [n] \times \mathbb{Z} \times [0,1]} e_i (X_j^{n,\varepsilon}(s-) - X_i^{n,\varepsilon}(s-)) g^{n,\varepsilon}(X^{n,\varepsilon}(s-); i, k, \theta) \mathrm{d}\pi^n(s, i, j, k, \theta), \qquad t \geq 0. \end{aligned} \tag{1}$$

Here e_i is the i-th standard basis vector of $\mathbb{R}^n$. Existence and uniqueness of a global strong solution are established in Lemma 4 below. The resulting process has paths in $\mathcal{D}([0, \infty), \mathbb{R}^n)$. Equivalently, we have

$$\begin{aligned} X_i^{n,\varepsilon}(t) = X_i^n(0) + W_i^n(t) + \int_0^t b(X_i^{n,\varepsilon}(s)) \mathrm{d}s \\ + \int_{(0,t] \times [n] \times \mathbb{Z} \times [0,1]} (X_j^{n,\varepsilon}(s-) - X_i^{n,\varepsilon}(s-)) g^{n,\varepsilon}(X^{n,\varepsilon}(s-); i, k, \theta) \mathrm{d}\pi_i^n(s, j, k, \theta), \qquad t \geq 0,\, i \in [n]. \end{aligned}$$

Here π_i^n is the restriction of π^n to marks whose removal label is i. Each accepted event therefore replaces the position of particle i by that of particle j, leaving the other particle positions unchanged. The normalization in $g^{n,\varepsilon}$ ensures that branching and selection events occur at total rate n, independently of the configuration:

$$\int_{(0,t] \times [n] \times [n] \times \mathbb{Z} \times [0,1]} g^{n,\varepsilon}(X^{n,\varepsilon}(s-); i, k, \theta) \hat{\pi}^n(\mathrm{d}s, \mathrm{d}i, \mathrm{d}j, \mathrm{d}k, \mathrm{d}\theta) = nt, \qquad t \geq 0. \tag{2}$$

The case $i = j$ is allowed. It corresponds to replacing the branching particle by its offspring and leaves the configuration unchanged. More generally, an accepted event produces a jump only when the two positions differ; the total rate n counts all accepted events.

Empirical and removal measures

We study the limit as $n \to \infty$, with $\varepsilon = \varepsilon_n \searrow 0$ along a suitable sequence. Let $\mathcal{M}_1$ denote the Polish space of Borel probability measures on $\mathbb{R}$, equipped with the topology of weak convergence. We introduce the empirical measure process

$$\mu^{n,\varepsilon;b,u_0} = \mu^{n,\varepsilon} : [0,\infty) \to \mathcal{M}_1, \quad t \mapsto \mu_t^{n,\varepsilon} = \frac{1}{n} \sum_{i \in [n]} \delta_{X_i^{n,\varepsilon;b,u_0}(t)}$$

associated with $X^{n,\varepsilon}$. Recalling $X^{n,\varepsilon} \in \mathcal{D}([0,\infty), \mathbb{R}^n)$, we have $\mu^{n,\varepsilon} \in \mathcal{D}([0,\infty), \mathcal{M}_1)$ a.s. We equip this path space with the usual Skorokhod J_1 topology and write time arguments as subscripts. For a path $\mu \in \mathcal{C}([0,\infty), \mathcal{M}_1)$, we call a measurable function $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ a density of μ if

$$\mu_t(\mathrm{d}x) = u(t,x)\mathrm{d}x, \qquad t \geq 0,$$

as an equality of measures.

To identify the limiting selection rule, we also keep track of where removals occur. Selection from the most populated cells should become removal at maxima of the limiting density. We therefore introduce a second component, which records the space-time locations of removal, normalized by the population size. At a time of removal t, its spatial distribution is uniform over the particles occupying the most populated cells immediately before t. Namely, define the Borel measure $\beta^{n,\varepsilon;b,u_0} = \beta^{n,\varepsilon}$ on $[0,\infty) \times \mathbb{R}$ by

$$\beta^{n,\varepsilon}(D) = \frac{1}{n} \int_{(0,\infty)\times[n]\times[n]\times\mathbb{Z}\times[0,1]} \mathbb{1}_{\{(s,X_i^{n,\varepsilon}(s-))\in D\}} g^{n,\varepsilon}(X^{n,\varepsilon}(s-); i,k,\theta)\mathrm{d}\hat{\pi}^n(s,i,j,k,\theta),$$

for Borel sets $D \in \mathcal{B}([0,\infty) \times \mathbb{R})$. This is the predictable compensator of the normalized removal measure. Its randomness comes from the configuration in the integrand. Let $\mathcal{M}_{\nearrow 1}$ be the set of Borel measures β on $[0,\infty) \times \mathbb{R}$ satisfying $\beta([0,T] \times \mathbb{R}) = T$ for every $T \geq 0$. Equivalently, these measures have Lebesgue measure as their time marginal. We endow this set with the topology of weak convergence on compact time horizons, i.e., the initial topology under the family $(\beta \mapsto \beta|_{[0,T]\times\mathbb{R}})_{T\geq 0}$ of measure restrictions. The resulting space is Polish. By (2), we have $\beta^{n,\varepsilon} \in \mathcal{M}_{\nearrow 1}$. For each $\beta \in \mathcal{M}_{\nearrow 1}$, the disintegration theorem gives a probability kernel $(\beta_t)_{t\geq 0}$ such that

$$\beta(\mathrm{d}t, \mathrm{d}x) = \beta_t(\mathrm{d}x)\mathrm{d}t.$$

The kernel is determined only for a.e. $t \geq 0$. When β is random, we choose a version measurable also in the underlying outcome, and the identity holds a.s.; see, e.g., [Kal17, Theorem 1.25].

The free obstacle problem

The macroscopic evolution equation combines diffusion, drift, branching and removal. To describe its structure as an obstacle problem, write $h(t) = \sup_{x\in\mathbb{R}} u(t,x)$ for all $t \geq 0$. The density lies below the upper obstacle $h(t)$, and removal acts only where this level is attained. Below the obstacle, the density therefore satisfies, in the sense of distributions,

$$\dot{u} = \tfrac{1}{2}u'' - (bu)' + u \qquad \text{on } \{u < h\},$$

with $\dot{u} = \partial_t u$ and $u' = \partial_x u$. The height $h(t)$ is not prescribed: It is determined together with the density under the unit mass constraint $\int_{\mathbb{R}} u(t,x)\mathrm{d}x = 1$. Following the terminology of [AMW26], we call this a parabolic free obstacle problem. We now make this description precise through a weak formulation.

First, to express the selection rule, for a function $u \in \mathcal{C}([0,\infty) \times \mathbb{R}, \mathbb{R})$ such that $u|_{[0,T]\times\mathbb{R}}$ is bounded for every $T < \infty$, set

$$\mathcal{B}^u = \{\beta \in \mathcal{M}_{\nearrow 1} : \beta(\{(t,x) \in [0,\infty) \times \mathbb{R} : u(t,x) < \|u(t,\cdot)\|_\infty\}) = 0\}.$$

For a nonnegative density u, this condition means that β_t is supported on the spatial maximizers of $u(t,\cdot)$ for a.e. t. For u as before and $\beta \in \mathcal{M}_{\nearrow 1}$, we call (u,β) a weak solution of

$$\dot{u} = \frac{1}{2}u'' - (bu)' + u - \beta, \qquad u(0,\cdot) = u_0, \qquad \beta \in \mathcal{B}^u, \tag{3}$$

if we have $\beta \in \mathcal{B}^u$ and the identity

$$\int_{\mathbb{R}} \varphi(x)(u(t,x) - u_0(x))\mathrm{d}x = \int_0^t \int_{\mathbb{R}} \left(\frac{1}{2}\varphi''(x) + \varphi'(x)b(x) + \varphi(x)\right) u(s,x)\mathrm{d}x\mathrm{d}s - \int_{[0,t]\times\mathbb{R}} \varphi(x)\mathrm{d}\beta(s,x)$$

holds for every $t \geq 0$ and every test function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$. In this equation, the diffusion term is $\frac{1}{2}u''$, transport is described by $-(bu)'$, branching contributes u, and β describes removal.

We work in the following class of continuous functions, whose members vanish at spatial infinity uniformly on bounded time intervals and have uniformly vanishing L^2-tails on those intervals. These conditions will be used in the regularization and uniqueness arguments:

$$\mathcal{U} = \left\{ u \in \mathcal{C}([0,\infty)\times\mathbb{R},\mathbb{R}) : \lim_{R\to\infty} \sup_{t\in[0,T]} \int_{[-R,R]^c} |u(t,x)|^2 \mathrm{d}x = 0 = \lim_{R\to\infty} \sup_{t\in[0,T]} \sup_{x\in[-R,R]^c} |u(t,x)| \text{ for all } T \geq 0 \right\}.$$

Stationary examples and motivating questions

To illustrate the role of the drift, recall two stationary examples from [AMW26] (rescaled to match the diffusion coefficient in (3)). For the smooth drift

$$b(x) = -\sqrt{2}\tanh(\sqrt{2}x),$$

set

$$w = \frac{\log(1+\sqrt{2})}{\sqrt{2}}, \qquad h = \frac{1}{2(w+1)}.$$

A stationary solution is given by

$$u(t,x) = \begin{cases} h, & |x| \leq w, \\ 2h\dfrac{\sinh(\sqrt{2}|x|)}{\cosh^2(\sqrt{2}|x|)}, & |x| > w, \end{cases}$$

and

$$\beta_t(\mathrm{d}x) = h\left(1 + \frac{2}{\cosh^2(\sqrt{2}x)}\right)\mathbb{1}_{[-w,w]}(x)\mathrm{d}x.$$

The density has a flat top on $[-w,w]$, and the removal measure is absolutely continuous and supported on it.

For the discontinuous inward drift

$$b(x) = -\sqrt{2}\,\mathrm{sign}(x),$$

a stationary solution is given by

$$u(t,x) = \frac{\sqrt{2}}{4}(1+\sqrt{2}|x|)\,\mathrm{e}^{-\sqrt{2}|x|}, \qquad \beta_t = \delta_0.$$

The density has a unique maximizer at 0, where the entire removal measure is concentrated. This example falls outside the scope of the well-posedness and convergence results in [AMW26], which require a Lipschitz drift.

These examples led to the questions posed in [AMW26, Problems 1.8 and 1.9]: whether well-posedness and convergence theory extends beyond Lipschitz drift, and how the regularity of the drift constrains the maximizer sets and the removal measure. In the stationary setting, we approach the latter question through an inverse problem, asking which removal measures can be realized by a suitable choice of drift.

Main Results

Under the standing assumptions on b and u_0, the following theorem establishes existence and uniqueness for (3) and identifies its solution as the deterministic hydrodynamic limit of the particle system. Convergence holds along a suitably chosen sequence of cell sizes $\varepsilon_n \searrow 0$. The path-space topology and the topology for the removal measures are those specified before.

Theorem 1. *There exists a null sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$ such that $(\mu^{n,\varepsilon_n}, \beta^{n,\varepsilon_n})_{n\in\mathbb{N}} \subseteq \mathcal{D}([0,\infty),\mathcal{M}_1)\times\mathcal{M}_{\nearrow 1}$ converges in probability to a deterministic limit $(\mu,\beta) \in \mathcal{C}([0,\infty),\mathcal{M}_1)\times\mathcal{M}_{\nearrow 1}$ such that μ has density $u \in \mathcal{U}$ and (u,β) is the unique solution of (3) in $\mathcal{U}\times\mathcal{M}_{\nearrow 1}$.*

For the inverse problem, we are given a target stationary removal measure β_0 and seek a drift and a stationary density that realize it. Assuming β_0 has compact support, we choose a density that is constant on an interval containing its support. In the interval, the diffusion term vanishes, and the stationary equation relates β_0 to the drift. We construct the drift using the distribution function of β_0, as specified in the following theorem.

Theorem 2. *Let $\beta_0 \in \mathcal{M}_1$ and $x_-, x_+ \in \mathbb{R}$ with $\beta_0([x_-,x_+]) = 1$. Setting $c = (\sqrt{2} + x_+ - x_-)^{-1}$, we consider the probability density $u_0 \in \mathcal{C}_0(\mathbb{R},\mathbb{R})$ given by*

$$u_0 : \mathbb{R}\to\mathbb{R}, \quad x \mapsto c\,\mathrm{e}^{-\sqrt{2}\,\mathrm{dist}(x,[x_-,x_+])},$$

and the càdlàg function

$$b : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} \sqrt{2}, & x < x_-, \\ \left(x - \frac{x_+ + x_-}{2}\right) + \frac{1 - 2\beta_0((-\infty, x])}{2c}, & x \in [x_-, x_+), \\ -\sqrt{2}, & x \geq x_+. \end{cases}$$

Finally, let $\mu \in \mathcal{C}([0,\infty), \mathcal{M}_1)$ have density $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, $(t,x) \mapsto u_0(x)$ and define $\beta = \lambda_{[0,\infty)} \otimes \beta_0 \in \mathcal{M}_{\nearrow 1}$. Then there exists a null sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$ such that $(\mu^{n,\varepsilon_n;b,u_0}, \beta^{n,\varepsilon_n;b,u_0})_{n\in\mathbb{N}} \subseteq \mathcal{D}([0,\infty), \mathcal{M}_1) \times \mathcal{M}_{\nearrow 1}$ converges in probability to (μ, β) and (u, β) is the unique solution of (3) in $\mathcal{U} \times \mathcal{M}_{\nearrow 1}$.

The constructed drift is bounded and càdlàg, hence satisfies our regularity assumption. Choosing β_0 to be the Cantor probability measure, for example, produces a stationary solution with a purely singular continuous removal measure. On $[0,1]$, the corresponding drift is an affine function minus a multiple of the Cantor function. Stationarity here refers to the PDE solution: $u(t,\cdot) = u_0$ and $\beta_t = \beta_0$ for a.e. $t \geq 0$.

Compact support is necessary within the stationary solution class considered here. A continuous probability density vanishing at infinity has a compact set of spatial maximizers, and the stationary removal measure is supported on that set. The inverse construction is not unique: for the same prescribed measure β_0, different choices of the containing interval $[x_-, x_+]$ yield different stationary densities and drifts. This is consistent with forward uniqueness in Theorem 1, where b and u_0 are fixed.

Related work

Branching particle systems with selection provide models for populations subject to reproduction and competition. In models motivated by natural selection, particle positions represent fitness, and selection favors individuals with higher fitness; see, e.g., [BDMM06]. A related model is the N-particle branching Brownian motion (N-BBM), in which each particle branches at rate one and the leftmost particle is removed upon each branching event, keeping the population size fixed. For suitable initial conditions, the fluctuations of the particle cloud on time scales of order $(\log N)^3$ were studied in [Mai16].

The hydrodynamic limit of the N-BBM is described by a free boundary problem, where the boundary represents the left edge of the population; see [DFPS19, BBP19]. The stationary empirical profile of the N-BBM, viewed from its leftmost particle, converges as $N \to \infty$ to the profile of the minimal travelling wave [BT24]. For suitable initial conditions, convergence to travelling waves and precise large time asymptotics for the front position of the free boundary problem were established in [BPT25]. Related connections between selection and free boundary problems arise for nonlocal branching models [DR11] and, in higher dimensions, for the Brownian bees model, where the particle farthest from the origin is removed upon each branching event [BBNP22]. Selection can also be encoded through a removal measure and a complementarity condition, without explicitly introducing a moving boundary. Such a weak formulation and its application to hydrodynamic limits were developed in [Ata25].

Spatial population models with mortality rates depending on local competition were studied in [FM04]. Different scalings yield convergence to a deterministic nonlinear integro-differential equation or to a nonlinear superprocess.

In [AMW26], branching random walks in arbitrary spatial dimension are regulated by keeping the population size fixed and removing particles from maximally populated sites. Our Brownian particle model leads to the same type of limiting free obstacle problem. The main technical contribution concerns the drift: We allow very general bounded drifts, whereas the well-posedness and convergence results of [AMW26] require Lipschitz continuity. Even in dimension one, admitting such drifts requires substantial changes to the regularity estimates and uniqueness argument.

Outline

In Section 2, we discuss some preliminaries and reduce Theorem 2 to Theorem 1. Section 3 deals with tightness while Section 4 establishes properties of arbitrary subsequential distributional limits. In Section 5, we prove that (3) has at most one solution in $\mathcal{U} \times \mathcal{M}_{\nearrow 1}$. Finally, in Section 6, the complementarity condition is verified and the proof of Theorem 1 is completed.

2 Preliminaries and Proof of Theorem 2

We start by showing the elementary fact that the total number of branching-removal events of the n particles on any bounded time interval I is Poisson distributed with parameter $n\lambda(I)$. Note that the number of jumps in that interval is bounded by the number of such events; in particular, removing the branching particle itself does not cause a jump.

Lemma 3. *Let $\varepsilon > 0$ and let Y be a predictable $\mathbb{R}^n$-valued process and $I \subseteq (0,\infty)$ a bounded interval. Then*

$$\int_{I\times[n]\times[n]\times\mathbb{Z}\times[0,1]} g^{n,\varepsilon}(Y(s);i,k,\theta)\mathrm{d}\pi^n(s,i,j,k,\theta)$$

is Poisson distributed with parameter $n\lambda(I)$.

Proof. Noting

$$\int_{I\times[n]\times[n]\times\mathbb{Z}\times[0,1]} \mathbb{1}_{\{\theta m^\varepsilon_{k'}(Y(s))|A^\varepsilon(Y(s))|\le 1,\, Y_{i'}(s)\in C^\varepsilon_{k'},\, k'\in A^\varepsilon(Y(s))\}}\mathrm{d}\hat\pi^n(s,i,j,k,\theta) = n\lambda(I) \quad \text{a.s.},$$

the classic formula for the Laplace transform of a Poisson integral with predictable integrand yields

$$\begin{aligned}
&\mathbb{E}\exp\left(-z\int_{I\times[n]\times[n]\times\mathbb{Z}\times[0,1]} g^{n,\varepsilon}(Y(s);i,k,\theta)\mathrm{d}\pi^n(s,i,j,k,\theta)\right)\\
&= \mathbb{E}\exp\left(-\int_{I\times[n]\times[n]\times\mathbb{Z}\times[0,1]}(1-\mathrm{e}^{-zg^{n,\varepsilon}(Y(s);i,k,\theta)})\mathrm{d}\hat\pi^n(s,i,j,k,\theta)\right)\\
&= \mathbb{E}\exp\left(-(1-\mathrm{e}^{-z})\int_{I\times[n]\times[n]\times\mathbb{Z}\times[0,1]} \mathbb{1}_{\{\theta m^\varepsilon_{k'}(Y(s))|A^\varepsilon(Y(s))|\le 1,\, Y_{i'}(s)\in C^\varepsilon_{k'},\, k'\in A^\varepsilon(Y(s))\}}\mathrm{d}\hat\pi^n(s,i,j,k,\theta)\right)\\
&= \exp(-(1-\mathrm{e}^{-z})n\lambda(I)), \qquad z\ge 0,
\end{aligned}$$

which is precisely the Laplace transform of a Poisson distribution with parameter $n\lambda(I)$. □

It is mathematical folklore that, due to boundedness of b, the jump diffusion $X^{n,\varepsilon}$ is well-defined by (1) for every $\varepsilon > 0$. In lack of a proper reference and for the sake of completeness, we briefly sketch the proof.

Lemma 4. *For every $\varepsilon > 0$, equation (1) has a unique global strong solution.*

Proof. Let $\tilde x \in \mathbb{R}^n$. According to a classic result [Ver80], the SDE

$$\tilde X^n(t) = \tilde x + W^n(t) + \int_0^t \big(b(\tilde X^n_i(s))\big)_{i\in[n]}\mathrm{d}s, \quad t\ge 0,$$

has a unique global strong solution $\tilde X^n$. As a consequence of Lemma 3, the number of jumps induced by the Poisson integral w.r.t. π^n in (1) is a.s. finite on any compact time interval. Thus, by applying the above findings piecewise, (1) must have a unique global strong solution as well. □

We end this section by deducing Theorem 2 from Theorem 1.

Proof of Theorem 2. By Theorem 1, there is a null sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$ such that $(\mu^{n,\varepsilon_n;b,u_0}, \beta^{n,\varepsilon_n;b,u_0})_{n\in\mathbb{N}}$ converges in probability to some deterministic $(\tilde\mu,\tilde\beta) \in \mathcal{C}([0,\infty),\mathcal{M}_1)\times\mathcal{M}_{\nearrow 1}$ satisfying the following properties: There exists a function $\tilde u \in \mathcal{U}$ with $\mathrm{d}\tilde\mu_t(x) = \tilde u(t,x)\mathrm{d}x$ for all $t\in[0,\infty)$ as equality of measures. Further, $(\tilde u,\tilde\beta)$ is the unique solution of (3) in $\mathcal{U}\times\mathcal{M}_{\nearrow 1}$. Noting $(u,\beta)\in\mathcal{U}\times\mathcal{M}_{\nearrow 1}$, it remains to prove that (u,β) is a solution of (3) as well. As a consequence of $\beta_0([x_-,x_+]) = 1$, we have $\beta\in\mathcal{B}^u$. Now let $\varphi\in\mathcal{C}^\infty_c(\mathbb{R},\mathbb{R})$. Since u and β are stationary in time, it suffices to prove

$$\int_{\mathbb{R}}\left(\frac{1}{2}\varphi''(x)+\varphi'(x)b(x)+\varphi(x)\right)u_0(x)\mathrm{d}x = \int_{\mathbb{R}}\varphi\mathrm{d}\beta_0 = \int_{[x_-,x_+]}\varphi\mathrm{d}\beta_0. \tag{4}$$

Inserting the definitions of b and u_0, we get

$$\begin{aligned}
\int_{-\infty}^{x_-}\left(\frac{1}{2}\varphi''(x)+\varphi'(x)b(x)+\varphi(x)\right)u_0(x)\mathrm{d}x &= \int_{-\infty}^{x_-}\left(\frac{1}{2}\varphi''(x)+\sqrt{2}\varphi'(x)+\varphi(x)\right)c\,\mathrm{e}^{\sqrt{2}(x-x_-)}\,\mathrm{d}x\\
&= \left[\left(\frac{1}{2}\varphi'(x)+\frac{1}{\sqrt{2}}\varphi(x)\right)c\,\mathrm{e}^{\sqrt{2}(x-x_-)}\right]_{x=-\infty}^{x_-}\\
&= \frac{c}{2}\varphi'(x_-)+\frac{\sqrt{2}c}{2}\varphi(x_-)
\end{aligned}$$

and similarly

$$\int_{x_+}^{\infty}\left(\frac{1}{2}\varphi''(x)+\varphi'(x)b(x)+\varphi(x)\right)u_0(x)\mathrm{d}x = -\frac{c}{2}\varphi'(x_+)+\frac{\sqrt{2}c}{2}\varphi(x_+).$$

Using integration by parts for Riemann and for Stieltjes integrals, we further obtain

$$\begin{aligned}
&\int_{x_-}^{x_+} \left(\frac{1}{2}\varphi''(x) + \varphi'(x)b(x) + \varphi(x)\right) u_0(x)\mathrm{d}x \\
&= \int_{x_-}^{x_+} c\left(\frac{1}{2}\varphi''(x) + \varphi'(x)\left(x - \frac{x_+ + x_-}{2} + \frac{1}{2c}\right) + \varphi(x)\right)\mathrm{d}x - \int_{x_-}^{x_+} \beta_0((-\infty, x])\varphi'(x)\mathrm{d}x \\
&= \frac{c}{2}[\varphi'(x)]_{x=x_-}^{x_+} + c\left[\varphi(x)\left(x - \frac{x_+ + x_-}{2} + \frac{1}{2c}\right)\right]_{x=x_-}^{x_+} - [\varphi(x)\beta_0((-\infty, x])]_{x=x_-}^{x_+} + \int_{[x_-, x_+]} \varphi \mathrm{d}\beta_0 \\
&= \frac{c}{2}\varphi'(x_+) - \frac{c}{2}\varphi'(x_-) + \frac{(x_+ - x_-)c}{2}(\varphi(x_+) + \varphi(x_-)) - \frac{1}{2}(\varphi(x_+) + \varphi(x_-)) + \int_{\mathbb{R}} \varphi \mathrm{d}\beta_0.
\end{aligned}$$

Summing the three equations and inserting the definition of c yields (4). □

3 Tightness

Let us now turn toward the proof of Theorem 1. We start with tightness. Some parts of our analysis are based on a natural embedding of the underlying particle system into the following pure branching particle system, without removals or transfers: For each $i \in [n]$, we independently start in $X_i^n(0)$ a binary branching process of unit branching rate, with particles moving (with independent paths after branching, as usual) according to the diffusion

$$\mathrm{d}Y(t) = \mathrm{d}W(t) + b(Y(t))\mathrm{d}t, \quad t \geq 0,$$

where W is a standard Brownian motion. Note that, by [Ver80], the above SDE has a unique global strong solution. Now fix $i \in [n]$. We denote the number of particles alive at time $t \geq 0$ in the i-th of these branching processes by $N_i^n(t)$. Further, we enumerate the particle trajectories of that branching process, understood as paths on the whole time axis $[0, \infty)$ (including the paths of the ancestors), and denote the resulting family of stochastic processes in $\mathcal{C}([0, \infty), \mathbb{R})$ by $(Y_{i,l}^n)_{l \in \mathbb{N}}$. W.l.o.g., we assume that the processes are enumerated such that $\{Y_{i,l}^n(t) : l \in [N_i^n(t)]\}$ is the set of all particle positions of the i-th branching process at time t for every $t \geq 0$. Interpreting the particle transfers in our original system as removals with simultaneous branching of another particle, there is a natural coupling between the two systems. For any given $\varepsilon > 0$, this coupling satisfies

$$\{(t, X_i^{n,\varepsilon}(t)) : t \geq 0,\, i \in [n]\} \subseteq \{(t, Y_{i,l}^n(t)) : t \geq 0,\, i \in [n],\, l \in [N_i^n(t)]\} \quad \text{a.s.}$$

We are now ready to state and prove our tightness result. Note that the cell size does not have to converge to 0.

Proposition 5. *The family $(\mu^{n,\varepsilon}, \beta^{n,\varepsilon})_{n \in \mathbb{N}, \varepsilon > 0}$ is tight. Further, for every sequence $(\varepsilon_n)_{n \in \mathbb{N}} \subseteq (0, \infty)$, each subsequential distributional limit μ of $(\mu^{n,\varepsilon_n})_{n \in \mathbb{N}}$ a.s. satisfies $\mu \in \mathcal{C}([0, \infty), \mathcal{M}_1)$.*

Since, for every sequence $(\varepsilon_n)_{n \in \mathbb{N}} \subseteq (0, \infty)$, the strong law of large numbers implies

$$\lim_{n \to \infty} \mu_0^{n,\varepsilon_n} = \lim_{n \to \infty} \frac{1}{n} \sum_{i \in [n]} \delta_{X_i^n(0)} = \mu_0 \quad \text{a.s.},$$

the notation in the proposition is consistent in the sense that the subsequential distributional limit μ a.s. starts in μ_0.

Proof of Proposition 5. As a consequence of Prohorov's theorem, tightness of $(\mu^{n,\varepsilon}, \beta^{n,\varepsilon})_{n \in \mathbb{N}, \varepsilon > 0}$ is equivalent to tightness of $(\mu^{n,\varepsilon_n}, \beta^{n,\varepsilon_n})_{n \in \mathbb{N}}$ for every sequence $(\varepsilon_n)_{n \in \mathbb{N}} \subseteq (0, \infty)$. So let $(\varepsilon_n)_{n \in \mathbb{N}} \subseteq (0, \infty)$. It suffices to separately establish tightness of $(\beta^{n,\varepsilon_n})_{n \in \mathbb{N}}$ and $\mathcal{C}$-tightness of $(\mu^{n,\varepsilon_n})_{n \in \mathbb{N}}$. In view of the topological structures of $\mathcal{D}([0, \infty), \mathcal{M}_1)$ and $\mathcal{M}_{\nearrow 1}$, it further suffices to establish these properties on any compact time horizon $[0, T]$. To this end, we fix $T > 0$ for the rest of the proof. To simplify notation, we will refrain from indicating restrictions explicitly, i.e., we will write μ^{n,ε_n} and β^{n,ε_n} instead of $\mu^{n,\varepsilon_n}|_{[0,T]}$ and $\beta^{n,\varepsilon_n}|_{[0,T] \times \mathbb{R}}$.

Let us start with tightness of $(\beta^{n,\varepsilon_n})_{n \in \mathbb{N}}$. To this end, let $\eta > 0$. For each $m \in \mathbb{N}$, choose $r_m > 0$ with

$$\mathbb{P}\left(\sup_{t \in [0,T]} |Y_{1,1}^1(t)| > r_m\right) \leq \frac{1}{m\, \mathrm{e}^T\, 2^m}.$$

Using the many-to-one lemma [HR17], the expected number of paths in the pure branching system which leave the ball of radius r_m around 0 before time T can be bounded by

$$\sum_{i\in[n]}\mathbb{E}\left[\sum_{l\in[N_i^n(T)]}\mathbb{1}_{\{\sup_{t\in[0,T]}|Y_{i,l}^n(t)|>r_m\}}\right]\le \mathrm{e}^T\sum_{i\in[n]}\mathbb{E}\mathbb{1}_{\{\sup_{t\in[0,T]}|Y_{i,1}^n(t)|>r_m\}}$$
$$= n\,\mathrm{e}^T\,\mathbb{P}\left(\sup_{t\in[0,T]}|Y_{1,1}^1(t)|>r_m\right)\le\frac{n}{m2^m},\qquad m\in\mathbb{N}. \tag{5}$$

Denoting by $\mathcal{M}_T([0,T]\times\mathbb{R})$ the space of all Borel measures on $[0,T]\times\mathbb{R}$ with total mass T, endowed with the topology of weak convergence, we set $O_m:=\{x\in\mathbb{R}:|x|>r_m\}$ and

$$K_m:=\left\{\beta\in\mathcal{M}_T([0,T]\times\mathbb{R}):\beta([0,T]\times O_m)\le\frac{1}{m}\right\},\quad m\in\mathbb{N}.$$

Let $m_0\in\mathbb{N}$ with $\sum_{m=m_0}^\infty\frac{1}{2^m}\le\eta$. Then $K:=\overline{\bigcap_{m=m_0}^\infty K_m}$ is compact by Prohorov's theorem. For the time being, fix $m\in\mathbb{N}$. Under the coupling described above, β^{n,ε_n} is concentrated on

$$\{(t,X_i^{n,\varepsilon_n}(t-)):t\in[0,T],\,i\in[n]\}\subseteq\{(t,Y_{i,l}^n(t-)):t\in[0,T],\,i\in[n],\,l\in[N_i^n(t)]\}.$$

Further, along any given trajectory of the pure branching system, a.s. at most one removal of a particle of the original system can happen. Thus the total number of removals at locations in O_m is bounded by the number of paths of the pure branching system which enter O_m. In other words, we have

$$\int_{(0,T]\times[n]\times[n]\times\mathbb{Z}\times[0,1]}\mathbb{1}_{\{|X_i^{n,\varepsilon_n}(s-)|>r_m\}}g^{n,\varepsilon_n}(X^{n,\varepsilon_n}(s-);i,k,\theta)\mathrm{d}\pi^n(s,i,j,k,\theta)$$
$$\le\sum_{i\in[n]}\sum_{l\in[N_i^n(T)]}\mathbb{1}_{\{\sup_{t\in[0,T]}|Y_{i,l}^n(t)|>r_m\}}\qquad\text{a.s.}$$

Together with (5), we obtain

$$\mathbb{E}[\beta^{n,\varepsilon_n}([0,T]\times O_m)]=\mathbb{E}\left[\frac{1}{n}\int_{(0,T]\times[n]\times[n]\times\mathbb{Z}\times[0,1]}\mathbb{1}_{\{|X_i^{n,\varepsilon_n}(s-)|>r_m\}}g^{n,\varepsilon_n}(X^{n,\varepsilon_n}(s-);i,k,\theta)\mathrm{d}\hat\pi^n(s,i,j,k,\theta)\right]$$
$$=\mathbb{E}\left[\frac{1}{n}\int_{(0,T]\times[n]\times[n]\times\mathbb{Z}\times[0,1]}\mathbb{1}_{\{|X_i^{n,\varepsilon_n}(s-)|>r_m\}}g^{n,\varepsilon_n}(X^{n,\varepsilon_n}(s-);i,k,\theta)\mathrm{d}\pi^n(s,i,j,k,\theta)\right]$$
$$\le\mathbb{E}\left[\frac{1}{n}\sum_{i\in[n]}\sum_{l\in[N_i^n(T)]}\mathbb{1}_{\{\sup_{t\in[0,T]}|Y_{i,l}^n(t)|>r_m\}}\right]\le\frac{1}{m2^m}.$$

We deduce

$$\mathbb{P}(\beta^{n,\varepsilon_n}\notin K)\le\sum_{m=m_0}^\infty\mathbb{P}\left(\beta^{n,\varepsilon_n}([0,T]\times O_m)>\frac{1}{m}\right)\le\sum_{m=m_0}^\infty m\mathbb{E}[\beta^{n,\varepsilon_n}([0,T]\times O_m)]\le\sum_{m=m_0}^\infty\frac{1}{2^m}\le\eta,\quad n\in\mathbb{N},$$

proving tightness of $(\beta^{n,\varepsilon_n})_{n\in\mathbb{N}}$.

Let us now turn toward $(\mu^{n,\varepsilon_n})_{n\in\mathbb{N}}$. We will invoke a classic characterization of $\mathcal{C}$-tightness. With the notation from above and

$$K_m':=\left\{\nu\in\mathcal{M}_1:\nu(O_m)\le\frac{1}{m}\right\},\quad m\in\mathbb{N},$$

the set $K':=\overline{\bigcap_{m=m_0}^\infty K_m'}$ is compact by Prohorov's theorem. For any $t\in[0,T]$ and $m\in\mathbb{N}$, the number of particles of the original system which are in O_m at time t is bounded by the number of paths of the pure branching system which enter O_m at all:

$$n\mu_t^{n,\varepsilon_n}(O_m)\le\sum_{i\in[n]}\sum_{l\in[N_i^n(T)]}\mathbb{1}_{\{\sup_{t\in[0,T]}|Y_{i,l}^n(t)|>r_m\}}.$$

Together with (5), we get

$$\sup_{t\in[0,T]}\mathbb{P}(\mu_t^{n,\varepsilon_n}\notin K')\le\sup_{t\in[0,T]}\sum_{m=m_0}^\infty\mathbb{P}\left(\mu_t^{n,\varepsilon_n}(O_m)>\frac{1}{m}\right)\le\sup_{t\in[0,T]}\sum_{m=m_0}^\infty\frac{m}{n}\mathbb{E}[n\mu_t^{n,\varepsilon_n}(O_m)]$$
$$\le\sum_{m=m_0}^\infty\frac{m}{n}\mathbb{E}\left[\sum_{i\in[n]}\sum_{l\in[N_i^n(T)]}\mathbb{1}_{\{\sup_{t\in[0,T]}|Y_{i,l}^n(t)|>r_m\}}\right]\le\sum_{m=m_0}^\infty\frac{1}{2^m}\le\eta,\qquad n\in\mathbb{N}. \tag{6}$$

Given $\delta > 0$, letting d_{LP} be the Lévy-Prohorov metric on $\mathcal{M}_1$ and setting

$$\Delta_\delta := \{(s,t) \in [0,T]^2 : 0 \le t - s \le \delta\},$$

we introduce moduli of continuity defined by

$$w_{\mathrm{LP}}(\mu, \delta) := \sup_{(s,t)\in\Delta_\delta} d_{\mathrm{LP}}(\mu_s, \mu_t), \quad \mu \in \mathcal{D}([0,T], \mathcal{M}_1),$$

as well as

$$w(f, \delta) := \sup_{(s,t)\in\Delta_\delta} |f(s) - f(t)|, \quad f \in \mathcal{D}([0,T], \mathbb{R}).$$

By continuity of $Y^1_{1,1}$, there must be a $\delta \in (0, \frac{\eta^3}{32(T+1)} \wedge 1)$ with $\mathbb{P}(w(Y^1_{1,1}, \delta) > \eta) \le \frac{\eta^2}{4}$. Denoting the η-neighborhood of a measurable set $D \in \mathcal{B}(\mathbb{R})$ by D^η, we obtain

$$\begin{aligned}
&\mathbb{P}(w_{\mathrm{LP}}(\mu^{n,\varepsilon_n}, \delta) > \eta) \\
&= \mathbb{P}(d_{\mathrm{LP}}(\mu^{n,\varepsilon_n}_t, \mu^{n,\varepsilon_n}_s) > \eta \text{ for some } (s,t) \in \Delta_\delta) \\
&\le \mathbb{P}(\mu^{n,\varepsilon_n}_s(D) > \mu^{n,\varepsilon_n}_t(D^\eta) + \eta \text{ or } \mu^{n,\varepsilon_n}_t(D) > \mu^{n,\varepsilon_n}_s(D^\eta) + \eta \text{ for some } (s,t) \in \Delta_\delta \text{ and } D \in \mathcal{B}(\mathbb{R})) \\
&\le \mathbb{P}(|\{i \in [n] : |X^{n,\varepsilon_n}_i(s) - X^{n,\varepsilon_n}_i(t)| > \eta\}| > \eta n \text{ for some } (s,t) \in \Delta_\delta) \\
&\le \mathbb{P}\left(|\{i \in [n] : |X^{n,\varepsilon_n}_i(s) - X^{n,\varepsilon_n}_i(t)| > \eta \text{ and } X^{n,\varepsilon_n}_i \in \mathcal{C}([s,t],\mathbb{R})\}| > \frac{\eta n}{2} \text{ for some } (s,t) \in \Delta_\delta\right) \\
&\qquad + \mathbb{P}\left(|\{i \in [n] : X^{n,\varepsilon_n}_i \notin \mathcal{C}([s,t],\mathbb{R})\}| > \frac{\eta n}{2} \text{ for some } (s,t) \in \Delta_\delta\right), \qquad n \in \mathbb{N}.
\end{aligned}$$

On the one hand, the choice of δ implies

$$\begin{aligned}
&\mathbb{P}\left(|\{i \in [n] : |X^{n,\varepsilon_n}_i(s) - X^{n,\varepsilon_n}_i(t)| > \eta \text{ and } X^{n,\varepsilon_n}_i \in \mathcal{C}([s,t],\mathbb{R})\}| > \frac{\eta n}{2} \text{ for some } (s,t) \in \Delta_\delta\right) \\
&\le \mathbb{P}\left(|\{i \in [n] : |X^{n,\varepsilon_n}_i(s) - X^{n,\varepsilon_n}_i(t)| > \eta \text{ and } X^{n,\varepsilon_n}_i \in \mathcal{C}([s,t],\mathbb{R}) \text{ for some } (s,t) \in \Delta_\delta\}| > \frac{\eta n}{2}\right) \\
&\le \frac{2}{\eta n}\mathbb{E}|\{i \in [n] : |X^{n,\varepsilon_n}_i(s) - X^{n,\varepsilon_n}_i(t)| > \eta \text{ and } X^{n,\varepsilon_n}_i \in \mathcal{C}([s,t],\mathbb{R}) \text{ for some } (s,t) \in \Delta_\delta\}| \\
&= \frac{2}{\eta n}\sum_{i\in[n]} \mathbb{P}(|X^{n,\varepsilon_n}_i(s) - X^{n,\varepsilon_n}_i(t)| > \eta \text{ and } X^{n,\varepsilon_n}_i \in \mathcal{C}([s,t],\mathbb{R}) \text{ for some } (s,t) \in \Delta_\delta) \\
&\le \frac{2}{\eta n}\sum_{i\in[n]} \mathbb{P}(|Y^1_{1,1}(s) - Y^1_{1,1}(t)| > \eta \text{ for some } (s,t) \in \Delta_\delta) = \frac{2}{\eta}\mathbb{P}(w(Y^1_{1,1}, \delta) > \eta) \le \frac{\eta}{2}, \qquad n \in \mathbb{N}.
\end{aligned}$$

On the other hand, bounding the number of particles which jump by the total number of jumps in the third step, applying Lemma 3 in the forth step and using the choice of δ in the final line, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{P}\left(|\{i \in [n] : X^{n,\varepsilon_n}_i \notin \mathcal{C}([s,t],\mathbb{R})\}| > \frac{\eta n}{2} \text{ for some } (s,t) \in \Delta_\delta\right) \\
&\le \limsup_{n\to\infty} \sum_{l\in[\lceil \frac{T}{\delta} \rceil]} \mathbb{P}\left(|\{i \in [n] : X^{n,\varepsilon_n}_i \notin \mathcal{C}([(l-1)\delta, l\delta],\mathbb{R})\}| > \frac{\eta n}{4}\right) \\
&\le \limsup_{n\to\infty} \frac{16}{\eta^2 n^2} \sum_{l\in[\lceil \frac{T}{\delta} \rceil]} \mathbb{E}|\{i \in [n] : X^{n,\varepsilon_n}_i \notin \mathcal{C}([(l-1)\delta, l\delta],\mathbb{R})\}|^2 \\
&\le \limsup_{n\to\infty} \frac{16}{\eta^2 n^2} \sum_{l\in[\lceil \frac{T}{\delta} \rceil]} \mathbb{E}\left(\int_{((l-1)\delta, l\delta]\times[n]\times[n]\times\mathbb{Z}\times[0,1]} g^{n,\varepsilon_n}(X^{n,\varepsilon_n}(s-); i, k, \theta) \mathrm{d}\pi^n(s,i,j,k,\theta)\right)^2 \\
&= \limsup_{n\to\infty} \frac{16}{\eta^2 n^2} \sum_{l\in[\lceil \frac{T}{\delta} \rceil]} (n\delta + n^2\delta^2) \le \frac{16(T+1)\delta}{\eta^2} \le \frac{\eta}{2}.
\end{aligned}$$

Combining the three chains of inequalities, we get

$$\limsup_{n\to\infty} \mathbb{P}(w_{\mathrm{LP}}(\mu^{n,\varepsilon_n}, \delta) > \eta) \le \eta.$$

By a classic characterization (cf. [EK86, Corollary VI.3.26] and [JS03, Proposition VI.3.26]), this estimate together with (6) precisely gives the desired $\mathcal{C}$-tightness of $(\mu^{n,\varepsilon_n})_{n\in\mathbb{N}}$. □

4 Properties of Subsequential Hydrodynamic Limits

We start by showing that any subsequential distributional limit μ of the particle distributions a.s. has a density in $\mathcal{U}$. Further, we show that this density is dominated by the particle density corresponding to the pure branching system introduced in Section 3. While we will only need the result for fixed cell size $\varepsilon > 0$, note that it is valid for any sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$. As usual, we denote the generic sample space encoding the underlying randomness by Ω.

Lemma 6. *Given $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$, let μ be a subsequential distributional limit of $(\mu^{n,\varepsilon_n})_{n\in\mathbb{N}}$. Then there exist a deterministic $v \in \mathcal{U}$, depending only on (u_0, b), and a product-measurable function*

$$u : [0,\infty) \times \mathbb{R} \times \Omega \to [0,\infty)$$

which a.s. satisfies $u \le v$ and $\mathrm{d}\mu_t(x) = u(t,x)\mathrm{d}x$ for all $t \ge 0$. Here, as usual, we suppress the third parameter of u, which is governing its randomness. Further, the function v may be chosen with $\int_{\mathbb{R}} v(t,x)\mathrm{d}x = \mathrm{e}^t$ for all $t \ge 0$ and with

$$\mathbb{E}\left[\int_0^t \int_{\mathbb{R}} g(s,x)\mathrm{d}\mu_s^{n,\varepsilon_n}(x)\mathrm{d}s\right] \le \int_0^t \int_{\mathbb{R}} g(s,x)v(s,x)\mathrm{d}x\mathrm{d}s, \quad n \in \mathbb{N},\, t \ge 0, \tag{7}$$

for every non-negative measurable function $g : [0,\infty) \times \mathbb{R} \to [0,\infty)$.

Proof. Given $Y(0) \sim \mu_0$, let $Y = (Y(t))_{t\ge 0}$ be the unique global strong solution to

$$\mathrm{d}Y(t) = \mathrm{d}W^1(t) + b(Y(t))\mathrm{d}t, \quad t \ge 0.$$

Then there exists a $\tilde{v} \in \mathcal{C}_b([0,\infty)\times\mathbb{R},\mathbb{R})$ with $\mathbb{P}(Y(t) \in D) = \int_D \tilde{v}(t,x)\mathrm{d}x$ for all $t \ge 0$ and $D \in \mathcal{B}(\mathbb{R})$. (In fact, the density $\tilde{v}$ is the unique solution to the Fokker-Planck equation associated with the above SDE.) Let $(Q_t)_{t\ge 0}$ be the transition semigroup of a standard Brownian motion and let $T \ge 0$. It is easy to check from [QZ02, Theorem 1] that there is a constant $C_T \ge 0$ with $\tilde{v}(t,\cdot) \le C_T Q_{2t}u_0$ for all $t \in [0,T]$. Since $[0,T] \to \mathcal{C}_0(\mathbb{R},\mathbb{R})$, $t \mapsto Q_{2t}u_0$ is continuous, a compactness argument yields $\lim_{R\to\infty} \sup_{t\in[0,T]} \sup_{x\in[-R,R]^c} |\tilde{v}(t,x)| = 0$. Moreover, we obtain

$$\begin{aligned}\sup_{t\in[0,T]} \int_{[-R,R]^c} |\tilde{v}(t,x)|^2\mathrm{d}x &\le \|\tilde{v}\|_\infty \sup_{t\in[0,T]} \mathbb{P}(|Y(t)| \ge R) \\ &\le \|\tilde{v}\|_\infty \mathbb{P}\left(|Y(0)| \ge \tfrac{R}{2}\right) + \|\tilde{v}\|_\infty \sup_{t\in[0,T]} \mathbb{P}\left(|W^1(t)| \ge \tfrac{R}{2} - \|b\|_\infty T\right) \xrightarrow{R\to\infty} 0.\end{aligned}$$

We deduce that $\tilde{v}$ and hence also $v : [0,\infty)\times\mathbb{R} \to \mathbb{R}$, $(t,x) \mapsto \mathrm{e}^t\,\tilde{v}(t,x)$ is an element of $\mathcal{U}$.

Now denote the space of finite Borel measures on $\mathbb{R}$, endowed with the topology of weak convergence, by $\mathcal{M}_f$. Using the notation for the pure branching system introduced in Section 3, let $\xi^n \in \mathcal{D}([0,\infty),\mathcal{M}_f)$ be its empirical measure process, defined by

$$\xi_t^n := \frac{1}{n}\sum_{i\in[n]} \sum_{l\in[N_i^n(t)]} \delta_{Y_{i,l}^n(t)}, \quad t \ge 0.$$

By classic branching process theory, $(\xi^n)_{n\in\mathbb{N}}$ converges in probability to a deterministic limit $\xi \in \mathcal{C}([0,\infty),\mathcal{M}_f)$, which satisfies $\mathrm{d}\xi_t(x) = v(t,x)\mathrm{d}x$ for all $t \ge 0$. Noting $\mu^{n,\varepsilon_n} \le \xi^n$ for every $n \in \mathbb{N}$ under the coupling described in Section 3, Strassen's theorem together with (subsequential) distributional convergence implies the existence of a coupling $(\tilde{\mu},\tilde{\xi})$ of the limits μ and ξ which satisfies $\tilde{\mu} \le \tilde{\xi}$ a.s. Here the order is preserved under convergence to continuous paths. In particular, we a.s. have $\mu_t \le \xi_t$ and hence $\mu_t \ll \lambda$ for all $t \ge 0$. Thus [DM82, Theorem V.58] and the subsequent remark guarantee the existence of a product-measurable $\breve{u} : [0,\infty)\times\mathbb{R}\times\Omega \to [0,\infty)$ which a.s. satisfies $\mathrm{d}\mu_t(x) = \breve{u}(t,x)\mathrm{d}x$ for all $t \ge 0$. Recalling $\mu_t \le \xi_t$ for all $t \ge 0$ a.s., the random function $u := \breve{u} \wedge v$ has the desired properties.

Finally, let $t \ge 0$ and $n \in \mathbb{N}$. Since $\tilde{v}(t,\cdot)$ is a probability density, we have $\int_{\mathbb{R}} v(t,x)\mathrm{d}x = \mathrm{e}^t$. Moreover, the coupling $\mu^{n,\varepsilon_n} \le \xi^n$ as well as the first-moment formula for the pure branching system and Tonelli's theorem give

$$\mathbb{E}\left[\int_0^t \int_{\mathbb{R}} g(s,x)\mathrm{d}\mu_s^{n,\varepsilon_n}(x)\mathrm{d}s\right] \le \mathbb{E}\left[\int_0^t \int_{\mathbb{R}} g(s,x)\mathrm{d}\xi_s^n(x)\mathrm{d}s\right] = \int_0^t \int_{\mathbb{R}} g(s,x)v(s,x)\mathrm{d}x\mathrm{d}s$$

for every non-negative measurable function $g : [0,\infty)\times\mathbb{R} \to [0,\infty)$. □

The next fundamental step is to establish a PDE, in weak form, for any subsequential distributional limit. We denote the partial derivative of a function $\varphi \in \mathcal{C}^{1,1}([0,\infty)\times\mathbb{R},\mathbb{R})$ w.r.t. its first component (representing time) by $\dot{\varphi}$ and the partial derivative w.r.t. its second component (representing space) by φ', similar to the notation used in the formal PDE (3).

Lemma 7. *Given $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$, let (μ,β) be a subsequential distributional limit of $(\mu^{n,\varepsilon_n},\beta^{n,\varepsilon_n})_{n\in\mathbb{N}}$. Then we a.s. have*

$$\begin{aligned}&\int_{\mathbb{R}} \varphi(t,x)\mathrm{d}\mu_t(x)\\ &= \int_{\mathbb{R}} \varphi(0,x)\mathrm{d}\mu_0(x) - \int_{[0,t]\times\mathbb{R}} \varphi(s,x)\mathrm{d}\beta(s,x) + \int_0^t \int_{\mathbb{R}} \left(\frac{1}{2}\varphi''(s,x) + \varphi'(s,x)b(x) + \varphi(s,x) + \dot{\varphi}(s,x)\right)\mathrm{d}\mu_s(x)\mathrm{d}s\end{aligned}$$

simultaneously for all $\varphi \in \mathcal{C}_c^{1,2}([0,\infty)\times\mathbb{R},\mathbb{R})$ and $t \geq 0$.

Proof. Let $\varphi \in \mathcal{C}_c^{1,2}([0,\infty)\times\mathbb{R},\mathbb{R})$ and $t \geq 0$ be fixed. Further, let $n \in \mathbb{N}$ and let $c \in [0,\infty)$ be a bound on $|\varphi|$ and $|\varphi'|$. We define

$$M^{n,1}(t) := \frac{1}{n}\sum_{i\in[n]} \int_0^t \varphi'(s, X_i^{n,\varepsilon_n}(s))\mathrm{d}W_i^n(s)$$

and

$$\begin{aligned}M^{n,2}(t) := &\frac{1}{n}\int_{(0,t]\times[n]\times[n]\times\mathbb{Z}\times[0,1]} (\varphi(s, X_j^{n,\varepsilon_n}(s-)) - \varphi(s, X_i^{n,\varepsilon_n}(s-)))g^{n,\varepsilon_n}(X^{n,\varepsilon_n}(s-);i,k,\theta)\mathrm{d}\pi^n(s,i,j,k,\theta)\\ &- \frac{1}{n}\int_{(0,t]\times[n]\times[n]\times\mathbb{Z}\times[0,1]} (\varphi(s, X_j^{n,\varepsilon_n}(s-)) - \varphi(s, X_i^{n,\varepsilon_n}(s-)))g^{n,\varepsilon_n}(X^{n,\varepsilon_n}(s-);i,k,\theta)\mathrm{d}\hat{\pi}^n(s,i,j,k,\theta).\end{aligned}$$

Observing

$$\begin{aligned}&\frac{1}{n}\int_{(0,t]\times[n]\times[n]\times\mathbb{Z}\times[0,1]} \varphi(s, X_j^{n,\varepsilon_n}(s-))g^{n,\varepsilon_n}(X^{n,\varepsilon_n}(s-);i,k,\theta)\mathrm{d}\hat{\pi}^n(s,i,j,k,\theta)\\ &= \int_0^t \left(\frac{1}{n}\sum_{j\in[n]} \varphi(s, X_j^{n,\varepsilon_n}(s-))\right)\left(\int_0^1 \sum_{i\in[n],k\in\mathbb{Z}} g^{n,\varepsilon_n}(X^{n,\varepsilon_n}(s-);i,k,\theta)\mathrm{d}\theta\right)\mathrm{d}s\\ &= \int_0^t \left(\frac{1}{n}\sum_{j\in[n]} \varphi(s, X_j^{n,\varepsilon_n}(s))\right)1\mathrm{d}s\\ &= \int_0^t \int_{\mathbb{R}} \varphi(s,x)\mathrm{d}\mu_s^{n,\varepsilon_n}(x)\mathrm{d}s\end{aligned}$$

and

$$\frac{1}{n}\int_{(0,t]\times[n]\times[n]\times\mathbb{Z}\times[0,1]} \varphi(s, X_i^{n,\varepsilon_n}(s-))g^{n,\varepsilon_n}(X^{n,\varepsilon_n}(s-);i,k,\theta)\mathrm{d}\hat{\pi}^n(s,i,j,k,\theta) = \int_{[0,t]\times\mathbb{R}} \varphi(s,x)\mathrm{d}\beta^{n,\varepsilon_n}(s,x),$$

Itô's formula for jump diffusions yields

$$\begin{aligned}&\int_{\mathbb{R}} \varphi(t,x)\mathrm{d}\mu_t^{n,\varepsilon_n}(x)\\ &= \frac{1}{n}\sum_{i\in[n]} \varphi(t, X_i^{n,\varepsilon_n}(t))\\ &= \frac{1}{n}\sum_{i\in[n]} \varphi(0, X_i^n(0)) + \frac{1}{n}\sum_{i\in[n]} \int_0^t \varphi'(s, X_i^{n,\varepsilon_n}(s))\mathrm{d}W_i^n(s) + \frac{1}{n}\sum_{i\in[n]} \int_0^t \frac{1}{2}\varphi''(s, X_i^{n,\varepsilon_n}(s))\mathrm{d}s\\ &\quad + \frac{1}{n}\sum_{i\in[n]} \int_0^t \dot{\varphi}(s, X_i^{n,\varepsilon_n}(s))\mathrm{d}s + \frac{1}{n}\sum_{i\in[n]} \int_0^t \varphi'(s, X_i^{n,\varepsilon_n}(s))b(X_i^{n,\varepsilon_n}(s))\mathrm{d}s\\ &\quad + \frac{1}{n}\int_{(0,t]\times[n]\times[n]\times\mathbb{Z}\times[0,1]} (\varphi(s, X_j^{n,\varepsilon_n}(s-)) - \varphi(s, X_i^{n,\varepsilon_n}(s-)))g^{n,\varepsilon_n}(X^{n,\varepsilon_n}(s-);i,k,\theta)\mathrm{d}\pi^n(s,i,j,k,\theta)\\ &= \int_{\mathbb{R}} \varphi(0,x)\mathrm{d}\mu_0^n(x) + \int_0^t \int_{\mathbb{R}} \left(\frac{1}{2}\varphi''(s,x) + \varphi(s,x) + \dot{\varphi}(s,x)\right)\mathrm{d}\mu_s^{n,\varepsilon_n}(x)\mathrm{d}s\\ &\quad + \int_0^t \int_{\mathbb{R}} \varphi'(s,x)b(x)\mathrm{d}\mu_s^{n,\varepsilon_n}(x)\mathrm{d}s - \int_{[0,t]\times\mathbb{R}} \varphi(s,x)\mathrm{d}\beta^{n,\varepsilon_n}(s,x) + M^{n,1}(t) + M^{n,2}(t).\end{aligned} \tag{8}$$

Noting that the integrand in the Poisson integral defining $M^{n,2}(t)$ is predictable, the (predictable) quadratic variations satisfy

$$\langle M^{n,1}\rangle(t) = \frac{1}{n^2}\sum_{i\in[n]} \int_0^t |\varphi'(s, X_i^{n,\varepsilon_n}(s))|^2\mathrm{d}s \leq \frac{c^2 t}{n} \xrightarrow{n\to\infty} 0$$

and

$$\langle M^{n,2}\rangle(t) = \frac{1}{n^2}\int_{(0,t]\times[n]\times[n]\times\mathbb{Z}\times[0,1]} (\varphi(s,X_j^{n,\varepsilon_n}(s-)) - \varphi(s,X_i^{n,\varepsilon_n}(s-)))^2 g^{n,\varepsilon_n}(X^{n,\varepsilon_n}(s-);i,k,\theta)\mathrm{d}\hat{\pi}^n(s,i,j,k,\theta)$$
$$\leq \frac{4c^2}{n^2}\int_{(0,t]\times[n]\times[n]\times\mathbb{Z}\times[0,1]} g^{n,\varepsilon_n}(X^{n,\varepsilon_n}(s-);i,k,\theta)\mathrm{d}\hat{\pi}^n(s,i,j,k,\theta) = \frac{4c^2t}{n} \xrightarrow{n\to\infty} 0.$$

Together with $M^{n,1}(0) = M^{n,2}(0) = 0$, this implies that $M^{n,1}(t)$ and $M^{n,2}(t)$ converge to 0 in probability as $n \to \infty$. Consequently, Slutsky's theorem implies that $(\mu, \beta, 0, 0)$ is a subsequential distributional limit of $(\mu^{n,\varepsilon_n}, \beta^{n,\varepsilon_n}, M^{n,1}(t), M^{n,2}(t))_{n\in\mathbb{N}}$. Invoking Skorokhod's representation theorem, we assume a.s. convergence. Since Proposition 5 guarantees $\mu \in \mathcal{C}([0,\infty),\mathcal{M}_1)$ a.s., we a.s. have convergence of $(\mu_s^{n,\varepsilon_n})_{n\in\mathbb{N}}$ to μ_s in $\mathcal{M}_1$ for all $s \geq 0$ along the considered subsequence, by classic properties of Skorokhod spaces. All integrands in (8) except the one involving b are bounded and continuous. For every bounded measurable function $\tilde{b} : \mathbb{R} \to \mathbb{R}$, set

$$I^n(\tilde{b}) := \int_0^t \int_{\mathbb{R}} \varphi'(s,x)\tilde{b}(x)\mathrm{d}\mu_s^{n,\varepsilon_n}(x)\mathrm{d}s, \qquad I(\tilde{b}) := \int_0^t \int_{\mathbb{R}} \varphi'(s,x)\tilde{b}(x)\mathrm{d}\mu_s(x)\mathrm{d}s.$$

Assume for a moment that $(I^n(b))_{n\in\mathbb{N}}$ converges to $I(b)$ in probability along the considered subsequence. Then taking the subsequential limit in probability in (8), with the help of the dominated convergence theorem, yields the claimed formula for any fixed φ and t. Since $\mathcal{C}_c^{1,2}([0,\infty)\times\mathbb{R},\mathbb{R})$ (w.r.t. its usual topology) and $[0,\infty)$ are separable, a continuity argument invoking the dominated convergence theorem guarantees the claimed uniformity of the a.s.-exceptional set.

It remains to prove the subsequential convergence of $(I^n(b))_{n\in\mathbb{N}}$ to $I(b)$. Let u and v be as in Lemma 6, also satisfying the additional properties mentioned there. Further, let $\rho \in \mathcal{C}_c^\infty(\mathbb{R},[0,\infty))$ with $\int_{\mathbb{R}}\rho(x)\mathrm{d}x = 1$ and $k \in \mathbb{N}$. Then $b_k := b * (k\rho(k\,\cdot))$ is continuous with $\|b_k\|_\infty \leq \|b\|_\infty$. By the subsequential convergence of μ_s^{n,ε_n} to μ_s for every $s \geq 0$ and the dominated convergence theorem, we have $I^n(b_k) \to I(b_k)$ a.s. as $n \to \infty$ along the considered subsequence. Noting $\int_0^t\int_{\mathbb{R}} v(s,x)\mathrm{d}x\mathrm{d}s = \mathrm{e}^t - 1$ and $\lim_{k\to\infty} b_k = b$ Lebesgue-a.e. by Lebesgue's differentiation theorem, (7) and the dominated convergence theorem imply

$$\sup_{n\in\mathbb{N}} \mathbb{E}|I^n(b) - I^n(b_k)| \leq \sup_{n\in\mathbb{N}} \mathbb{E}\left[\int_0^t\int_{\mathbb{R}} c|b(x)-b_k(x)|\mathrm{d}\mu_s^{n,\varepsilon_n}(x)\mathrm{d}s\right] \leq \int_0^t\int_{\mathbb{R}} c|b(x)-b_k(x)|v(s,x)\mathrm{d}x\mathrm{d}s \xrightarrow{k\to\infty} 0.$$

Invoking properties of u, we get

$$|I(b_k) - I(b)| \leq \int_0^t\int_{\mathbb{R}} c|b_k(x)-b(x)|u(s,x)\mathrm{d}x\mathrm{d}s \leq \int_0^t\int_{\mathbb{R}} c|b_k(x)-b(x)|v(s,x)\mathrm{d}x\mathrm{d}s \xrightarrow{k\to\infty} 0 \quad \text{a.s.}$$

Using the triangle inequality and Markov's inequality, the desired convergence in probability follows by first letting $n \to \infty$ along the considered subsequence and then $k \to \infty$. □

We end this section by essentially showing uniform equicontinuity of all possible subsequential particle densities.

Lemma 8. *Let $\beta \in \mathcal{M}_{\nearrow 1}$ and let $u : [0,\infty)\times\mathbb{R} \to \mathbb{R}$ be measurable with*

$$\int_{\mathbb{R}} \varphi(t,x)u(t,x)\mathrm{d}x$$
$$= \int_{\mathbb{R}} \varphi(0,x)u_0(x)\mathrm{d}x - \int_{[0,t]\times\mathbb{R}} \varphi(s,x)\mathrm{d}\beta(s,x)$$
$$+ \int_0^t\int_{\mathbb{R}} \left(\frac{1}{2}\varphi''(s,x) + \varphi'(s,x)b(x) + \varphi(s,x) + \dot{\varphi}(s,x)\right)u(s,x)\mathrm{d}x\mathrm{d}s, \qquad \varphi \in \mathcal{C}_c^\infty([0,\infty)\times\mathbb{R},\mathbb{R}),\ t \geq 0.$$

Assume that, for every $T \geq 0$, there is a constant $C_T \in [0,\infty)$ with $\|u|_{[0,T]\times\mathbb{R}}\|_\infty \leq C_T$. Then there exists a function $\tilde{u} \in \mathcal{C}([0,\infty)\times\mathbb{R},\mathbb{R})$ with $\tilde{u}(t,x) = u(t,x)$ for a.e. $x \in \mathbb{R}$ and all $t \geq 0$. Further, for every $T \geq 0$, the restriction $\tilde{u}|_{[0,T]\times\mathbb{R}}$ has a modulus of continuity, which depends only on $(u_0, \|b\|_\infty, T, C_T)$, but not on β or u itself.

Proof. To simplify notation, we write $u_s := u(s,\cdot)$ for $s > 0$. Let $(Q_t)_{t\geq 0}$ be the transition semigroup of a standard Brownian motion. For the time being, we fix $t > 0$ and $\psi \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{R})$. Let $\varphi \in \mathcal{C}_b^\infty([0,\infty)\times\mathbb{R},\mathbb{R})$ be a function satisfying $\varphi(s,x) := \mathrm{e}^{t-s}\,Q_{t-s}\psi(x)$ for all $(s,x) \in [0,t]\times\mathbb{R}$. Further, let $R > 0$ and $\chi_R \in \mathcal{C}_c^\infty(\mathbb{R},[0,1])$ with $\chi_R|_{[-R,R]} = 1$ as well as $\|\chi_R'\|_\infty \leq 1$ and $\|\chi_R''\|_\infty \leq 1$. Then $\varphi_R : [0,\infty)\times\mathbb{R} \to \mathbb{R}$, $(s,x) \mapsto \chi_R(x)\varphi(s,x)$ satisfies $\varphi_R \in \mathcal{C}_c^\infty([0,\infty)\times\mathbb{R},\mathbb{R})$. Inserting this test function into the given PDE in weak form and using the

PDE $\dot\varphi = -\frac{1}{2}\varphi'' - \varphi$ on $[0,t]\times\mathbb{R}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \chi_R(x)\psi(x)u(t,x)\mathrm{d}x \\
&= \int_{\mathbb{R}} \varphi_R(0,x)u_0(x)\mathrm{d}x - \int_{[0,t]\times\mathbb{R}} \varphi_R(s,x)\mathrm{d}\beta(s,x) \\
&\quad + \int_0^t \int_{\mathbb{R}} \left(\frac{1}{2}\varphi_R''(s,x) + \varphi_R'(s,x)b(x) - \frac{1}{2}\chi_R(x)\varphi''(s,x)\right) u_s(x)\mathrm{d}x\mathrm{d}s \\
&= \int_{\mathbb{R}} \chi_R(x)\varphi(0,x)u_0(x)\mathrm{d}x - \int_{[0,t]\times\mathbb{R}} \chi_R(x)\varphi(s,x)\mathrm{d}\beta(s,x) \\
&\quad + \int_0^t \int_{\mathbb{R}} \left(\frac{1}{2}\chi_R''(x)\varphi(s,x) + \chi_R'(x)\varphi'(s,x) + \chi_R'(x)\varphi(s,x)b(x) + \chi_R(x)\varphi'(s,x)b(x)\right) u_s(x)\mathrm{d}x\mathrm{d}s. \qquad (9)
\end{aligned}$$

Since $(Q_t)_{t\geq 0}$ is a contraction semigroup on $L^1(\mathbb{R},\mathbb{R})$, the functions $\varphi|_{[0,t]\times\mathbb{R}}$ and $\varphi'|_{[0,t]\times\mathbb{R}}$ are integrable. Applying the dominated convergence theorem, we deduce

$$\lim_{R\to\infty} \int_0^t \int_{\mathbb{R}} \left(\frac{1}{2}\chi_R''(x)\varphi(s,x) + \chi_R'(x)\varphi'(s,x) + \chi_R'(x)\varphi(s,x)b(x)\right) u_s(x)\mathrm{d}x\mathrm{d}s = 0.$$

Since Q_t is self-adjoint (or using Fubini's theorem), we get

$$\int_{\mathbb{R}} \chi_R(x)\varphi(0,x)u_0(x)\mathrm{d}x = \int_{\mathbb{R}} \mathrm{e}^t\, Q_t\psi(x)\chi_R(x)u_0(x)\mathrm{d}x = \int_{\mathbb{R}} \mathrm{e}^t\, \psi(x)Q_t(\chi_R u_0)(x)\mathrm{d}x \xrightarrow{R\to\infty} \int_{\mathbb{R}} \mathrm{e}^t\, \psi(x)Q_t u_0(x)\mathrm{d}x,$$

where the last step follows from the fact that the dominated convergence theorem implies

$$\|Q_t((1-\chi_R)u_0)\|_1 \leq \|(1-\chi_R)u_0\|_1 \xrightarrow{R\to\infty} 0.$$

Together with integration by parts, a similar argument yields

$$\begin{aligned}
\lim_{R\to\infty} \int_0^t \int_{\mathbb{R}} \chi_R(x)\varphi'(s,x)b(x)u_s(x)\mathrm{d}x\mathrm{d}s &= \lim_{R\to\infty} \int_0^t \int_{\mathbb{R}} \mathrm{e}^{t-s}\, Q_{t-s}\psi'(x)(\chi_R b u_s)(x)\mathrm{d}x\mathrm{d}s \\
&= \lim_{R\to\infty} \int_0^t \int_{\mathbb{R}} \mathrm{e}^{t-s}\, \psi'(x)Q_{t-s}(\chi_R b u_s)(x)\mathrm{d}x\mathrm{d}s \\
&= \int_0^t \int_{\mathbb{R}} \mathrm{e}^{t-s}\, \psi'(x)Q_{t-s}(b u_s)(x)\mathrm{d}x\mathrm{d}s \\
&= -\int_0^t \int_{\mathbb{R}} \mathrm{e}^{t-s}\, \psi(x)(Q_{t-s}(b u_s))'(x)\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Consequently, taking the limit $R\to\infty$ in (9), we obtain

$$\int_{\mathbb{R}} \psi(x)u(t,x)\mathrm{d}x = \int_{\mathbb{R}} \mathrm{e}^t\, \psi(x)Q_t u_0(x)\mathrm{d}x - \int_{[0,t]\times\mathbb{R}} \varphi(s,x)\mathrm{d}\beta(s,x) - \int_0^t \int_{\mathbb{R}} \mathrm{e}^{t-s}\, \psi(x)(Q_{t-s}(b u_s))'(x)\mathrm{d}x\mathrm{d}s. \qquad (10)$$

Now let $(q_t)_{t>0}$ be the family of transition densities of a standard Brownian motion. By Fubini's theorem, we have

$$\int_{[0,t]\times\mathbb{R}} \varphi(s,x)\mathrm{d}\beta(s,x) = \int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{t-s}\, q_{t-s}(x-y)\psi(y)\mathrm{d}y\mathrm{d}\beta_s(x)\mathrm{d}s = \int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{t-s}\, q_{t-s}(y-x)\mathrm{d}\beta_s(y)\psi(x)\mathrm{d}x\mathrm{d}s.$$

Inserting this into (10) and recalling that the resulting equation holds for all $\psi\in\mathcal{C}_c^\infty(\mathbb{R},\mathbb{R})$, we get $u(t,\cdot) = \tilde u(t,\cdot)$ λ-a.e. for

$$\tilde u : [0,\infty)\times\mathbb{R}\to\mathbb{R}, \quad (t,x)\mapsto \mathrm{e}^t\, Q_t u_0(x) - \int_0^t \int_{\mathbb{R}} \mathrm{e}^{t-s}\, q_{t-s}(y-x)\mathrm{d}\beta_s(y)\mathrm{d}s - \int_0^t \mathrm{e}^{t-s}(Q_{t-s}(b u_s))'(x)\mathrm{d}s.$$

Since $u_0\in\mathcal{C}_0(\mathbb{R},\mathbb{R})$ is uniformly continuous, standard results imply that $[0,\infty)\times\mathbb{R}\to\mathbb{R}$, $(t,x)\mapsto Q_t u_0(x)$ is uniformly continuous. Consequently, the first term in the definition of $\tilde u$ has the desired continuity property. Regarding the remaining two terms, it suffices by the triangle inequality to consider the two coordinate directions separately and establish the existence of uniform moduli of continuity. Let $T\geq 0$. We start with continuity in space. Let $t\in[0,T]$ as well as $\delta>0$ and $x,z\in\mathbb{R}$ with $|x-z|=\delta$. We have

$$\begin{aligned}
&\left|\int_0^t \int_{\mathbb{R}} \mathrm{e}^{t-s}\, q_{t-s}(y-x)\mathrm{d}\beta_s(y)\mathrm{d}s - \int_0^t \int_{\mathbb{R}} \mathrm{e}^{t-s}\, q_{t-s}(y-z)\mathrm{d}\beta_s(y)\mathrm{d}s\right| \\
&\leq \mathrm{e}^T \int_0^t \sup_{\tilde y\in\mathbb{R}} |q_{t-s}(\tilde y) - q_{t-s}(\tilde y+\delta)|\mathrm{d}s \leq \mathrm{e}^T \int_0^t \min\{\delta\|q_{t-s}'\|_\infty, 2\|q_{t-s}\|_\infty\}\mathrm{d}s \leq \mathrm{e}^T \int_0^t \min\left\{\frac{\delta}{t-s}, \frac{2}{\sqrt{t-s}}\right\}\mathrm{d}s.
\end{aligned}$$

Noting

$$\int_{\mathbb{R}} |q'_{t-s}(\tilde{y}) - q'_{t-s}(\tilde{y}+\delta)| \mathrm{d}\tilde{y} \le \min\{\delta \|q''_{t-s}\|_1, 2\|q'_{t-s}\|_1\} \le \min\left\{\frac{\delta}{t-s}, \frac{2}{\sqrt{t-s}}\right\}, \quad s \in (0,t],$$

we also get

$$\begin{aligned}\left|\int_0^t \mathrm{e}^{t-s}(Q_{t-s}(bu_s))'(x)\mathrm{d}s - \int_0^t \mathrm{e}^{t-s}(Q_{t-s}(bu_s))'(z)\mathrm{d}s\right| &= \left|\int_0^t \mathrm{e}^{t-s}\int_{\mathbb{R}}(q'_{t-s}(x-y) - q'_{t-s}(z-y))b(y)u_s(y)\mathrm{d}y\mathrm{d}s\right| \\ &\le \mathrm{e}^T \|b\|_\infty C_T \int_0^t \min\left\{\frac{\delta}{t-s}, \frac{2}{\sqrt{t-s}}\right\}\mathrm{d}s.\end{aligned}$$

It remains to observe that the dominated convergence theorem implies

$$\lim_{\delta \searrow 0} \sup_{t\in[0,T]} \int_0^t \min\left\{\frac{\delta}{t-s}, \frac{2}{\sqrt{t-s}}\right\}\mathrm{d}s = \lim_{\delta\searrow 0}\int_0^T \min\left\{\frac{\delta}{r}, \frac{2}{\sqrt{r}}\right\}\mathrm{d}r = 0.$$

We now turn toward continuity in space. Let $x \in \mathbb{R}$ as well as $\delta \in (0,T]$ and $t \in [0, T-\delta]$. Noting $|\partial_r q_r(y)| \le \frac{1}{\sqrt{r^3}}$ and thus, by the mean value theorem,

$$|\,\mathrm{e}^\delta q_{r+\delta}(y) - q_r(y)| \le (\mathrm{e}^\delta - 1)q_{r+\delta}(y) + |q_{r+\delta}(y) - q_r(y)| \le \min\left\{\frac{\mathrm{e}^\delta - 1}{\sqrt{r}} + \frac{\delta}{\sqrt{r^3}}, \frac{\mathrm{e}^\delta + 1}{\sqrt{r}}\right\}, \quad r > 0,\, y \in \mathbb{R},$$

we get

$$\begin{aligned}&\left|\int_0^{t+\delta}\int_{\mathbb{R}} \mathrm{e}^{t+\delta-s}\, q_{t+\delta-s}(y-x)\mathrm{d}\beta_s(y)\mathrm{d}s - \int_0^t\int_{\mathbb{R}} \mathrm{e}^{t-s}\, q_{t-s}(y-x)\mathrm{d}\beta_s(y)\mathrm{d}s\right| \\ &\le \mathrm{e}^T \int_t^{t+\delta} \frac{1}{\sqrt{t+\delta-s}}\mathrm{d}s + \mathrm{e}^T \int_0^t \sup_{y\in\mathbb{R}} |\,\mathrm{e}^\delta q_{t+\delta-s}(y) - q_{t-s}(y)|\mathrm{d}s \\ &\le \mathrm{e}^T \int_0^\delta \frac{1}{\sqrt{r}}\mathrm{d}r + \mathrm{e}^T \int_0^T \min\left\{\frac{\mathrm{e}^\delta - 1}{\sqrt{r}} + \frac{\delta}{\sqrt{r^3}}, \frac{\mathrm{e}^\delta + 1}{\sqrt{r}}\right\}\mathrm{d}r,\end{aligned}$$

which converges to 0 as $\delta \searrow 0$ by the dominated convergence theorem. Using $\|q'_r\|_1 \le \frac{1}{\sqrt{r}}$ and $\|\partial_r q'_r\|_1 \le \frac{2}{\sqrt{r^3}}$, the mean value theorem again yields

$$\|\,\mathrm{e}^\delta q'_{r+\delta} - q'_r\|_1 \le (\mathrm{e}^\delta - 1)\|q'_{r+\delta}\|_1 + \|q'_{r+\delta} - q'_r\|_1 \le \min\left\{\frac{\mathrm{e}^\delta - 1}{\sqrt{r}} + \frac{2\delta}{\sqrt{r^3}}, \frac{\mathrm{e}^\delta + 1}{\sqrt{r}}\right\}, \quad r > 0.$$

We finally obtain

$$\begin{aligned}&\left|\int_0^{t+\delta} \mathrm{e}^{t+\delta-s}(Q_{t+\delta-s}(bu_s))'(x)\mathrm{d}s - \int_0^t \mathrm{e}^{t-s}(Q_{t-s}(bu_s))'(x)\mathrm{d}s\right| \\ &= \left|\int_0^{t+\delta}\int_{\mathbb{R}} \mathrm{e}^{t+\delta-s}\, q'_{t+\delta-s}(x-y)b(y)u_s(y)\mathrm{d}y\mathrm{d}s - \int_0^t\int_{\mathbb{R}} \mathrm{e}^{t-s}\, q'_{t-s}(x-y)b(y)u_s(y)\mathrm{d}y\mathrm{d}s\right| \\ &\le \mathrm{e}^T \|b\|_\infty C_T \left(\int_t^{t+\delta} \|q'_{t+\delta-s}\|_1\mathrm{d}s + \int_0^t \|\,\mathrm{e}^\delta q'_{t+\delta-s} - q'_{t-s}\|_1\mathrm{d}s\right) \\ &\le \mathrm{e}^T \|b\|_\infty C_T \left(\int_0^\delta \frac{1}{\sqrt{r}}\mathrm{d}r + \int_0^T \min\left\{\frac{\mathrm{e}^\delta - 1}{\sqrt{r}} + \frac{2\delta}{\sqrt{r^3}}, \frac{\mathrm{e}^\delta + 1}{\sqrt{r}}\right\}\mathrm{d}r\right)\end{aligned}$$

which again converges to 0 as $\delta \searrow 0$. □

5 Uniqueness of PDE Solution

Our proof builds on the uniqueness argument of [AMW26, Section 5]. A key difficulty in extending that argument to discontinuous drifts is that its estimates rely on regularity of b. We overcome this difficulty by controlling the contribution of the drift using only its boundedness, and obtain the following result.

Proposition 9. *In the space $\mathcal{U} \times \mathcal{M}_{\nearrow 1}$, the PDE problem* (3) *has at most one solution.*

Throughout this section, let $t > 0$ be fixed, $\eta \in \mathcal{C}_c^\infty(\mathbb{R}, [0,\infty))$ a symmetric mollifier supported on $[-1,1]$, and $\delta > 0$. Further, we use the following notation: Introducing $\eta_\delta : \mathbb{R} \to [0,\infty)$, $x \mapsto \frac{1}{\delta}\eta(\frac{x}{\delta})$, we set $\psi^\delta := \eta_\delta * \psi$ for $\psi : \mathbb{R} \to \mathbb{R}$. For $\nu \in \mathcal{M}_1$, we denote the smooth Lebesgue density of $\eta_\delta * \nu$ by ν^δ. Finally, we write $w_s := w(s,\cdot)$ for $w \in \mathcal{U}$ and $s \in [0,t]$ to simplify the notation.

We start with an auxiliary result stating that, for any $u \in \mathcal{U}$, certain mollifications in space converge locally uniformly in time.

Lemma 10. *Let $w \in \mathcal{U}$. Then we have*

$$\lim_{\delta \searrow 0} \sup_{s\in[0,t]} \|w_s - (w_s^\delta)^\delta\|_\infty = 0 \quad \textit{and} \quad \lim_{\delta \searrow 0} \sup_{s\in[0,t]} \|w_s - w_s^\delta\|_2 = 0.$$

Proof. Regarding the first claim, it is straightforward to check that $w|_{[0,t]\times\mathbb{R}}$ is uniformly continuous: On the one hand, for every $\varepsilon > 0$, there exists an $R > 0$ with $\sup_{s\in[0,t]} \sup_{x\in[-R,R]^c} |w_s(x)| < \varepsilon$. On the other hand, w is uniformly continuous on the compact set $[0,t] \times [-R,R]$. Noting $\int_\mathbb{R} \eta_\delta(x)\mathrm{d}x = 1$ and $\mathrm{supp}(\eta_\delta) \subseteq [-\delta,\delta]$, we get

$$\begin{aligned}
|w_s(x) - (w_s^\delta)^\delta(x)| &= \left| \int_\mathbb{R}\int_\mathbb{R} \eta_\delta(x-y)\eta_\delta(y-z)(w_s(x) - w_s(z))\mathrm{d}z\mathrm{d}y \right| \\
&\le \int_{x-\delta}^{x+\delta}\int_{y-\delta}^{y+\delta} \eta_\delta(x-y)\eta_\delta(y-z)|w_s(x) - w_s(z)|\mathrm{d}z\mathrm{d}y \\
&\le \sup_{z\in[x-2\delta,x+2\delta]} |w_s(x) - w_s(z)|, \qquad x \in \mathbb{R},\, s \in [0,t].
\end{aligned}$$

As $\delta \searrow 0$, this converges to 0 uniformly in $x \in \mathbb{R}$ and $s \in [0,t]$.

Let us now turn toward the second claim. Let $\varepsilon > 0$. As a consequence of $w \in \mathcal{U}$, there exists an $R > 1$ with $\sup_{s\in[0,t]} \int_{[-(R-1),R-1]^c} |w_s(x)|^2\mathrm{d}x < \frac{\varepsilon}{8}$. We obtain

$$\begin{aligned}
&\int_{[-R,R]^c}\int_{-\delta}^{\delta} |w_s(x-y) - w_s(x)|^2\eta_\delta(y)\mathrm{d}y\mathrm{d}x \\
&\le 2\int_{[-R,R]^c}\int_{-\delta}^{\delta} |w_s(x-y)|^2\eta_\delta(y)\mathrm{d}y\mathrm{d}x + 2\int_{[-R,R]^c}\int_{-\delta}^{\delta} |w_s(x)|^2\eta_\delta(y)\mathrm{d}y\mathrm{d}x \\
&\le 4\int_{[-(R-1),R-1]^c} |w_s(x)|^2\mathrm{d}x \int_{-\delta}^{\delta} \eta_\delta(y)\mathrm{d}y < \frac{\varepsilon}{2}, \qquad s \in [0,t],\, \delta \in (0,1].
\end{aligned}$$

Since $w|_{[0,t]\times[-(R+1),R+1]}$ is uniformly continuous, there exists a $\delta_0 \in (0,1]$ with $|w_s(x) - w_s(y)| < \sqrt{\frac{\varepsilon}{4R}}$ for all $s \in [0,t]$ and all $x, y \in [-(R+1), R+1]$ with $|x-y| < \delta_0$. We deduce

$$\int_{-R}^{R}\int_{-\delta}^{\delta} |w_s(x-y) - w_s(x)|^2\eta_\delta(y)\mathrm{d}y\mathrm{d}x < \int_{-R}^{R} \frac{\varepsilon}{4R}\mathrm{d}x \int_{-\delta}^{\delta} \eta_\delta(y)\mathrm{d}y = \frac{\varepsilon}{2}, \quad s \in [0,t],\, \delta \in (0,\delta_0].$$

Combining Jensen's inequality, applied to the probability measure with density η_δ, with the two bounds we just established, we finally get

$$\begin{aligned}
\|w_s - w_s^\delta\|_2^2 &= \int_\mathbb{R} \left| \int_{-\delta}^{\delta} (w_s(x-y) - w_s(x))\eta_\delta(y)\mathrm{d}y \right|^2 \mathrm{d}x \\
&\le \int_\mathbb{R}\int_{-\delta}^{\delta} |w_s(x-y) - w_s(x)|^2\eta_\delta(y)\mathrm{d}y\mathrm{d}x < \varepsilon, \qquad s \in [0,t],\, \delta \in (0,\delta_0],
\end{aligned}$$

proving $\lim_{\delta\searrow 0} \sup_{s\in[0,t]} \|w_s - w_s^\delta\|_2 = 0$. □

Proof of Proposition 9. Let $(u,\beta), (\tilde{u},\tilde{\beta}) \in \mathcal{U} \times \mathcal{M}_{\nearrow 1}$ be two solutions to (3). Let us start by briefly arguing that it suffices to show $u = \tilde{u}$. Indeed, by definition of a solution, $u = \tilde{u}$ implies

$$\int_{[0,t]\times\mathbb{R}} \varphi(x)\mathrm{d}\beta(s,x) = \int_{[0,t]\times\mathbb{R}} \varphi(x)\mathrm{d}\tilde{\beta}(s,x), \quad t \ge 0,\, \varphi \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{R}).$$

Recalling the definition of $\mathcal{M}_{\nearrow 1}$ and invoking standard arguments from measure theory, this, in turn, yields $\beta = \tilde{\beta}$.

Now let $v := u - \tilde{u} \in \mathcal{U}$ and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{R})$. The definition of a solution, applied to $\varphi^\delta \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{R})$, yields

$$\int_\mathbb{R} \varphi^\delta(u_t - u_0)\mathrm{d}\lambda = \int_0^t \int_\mathbb{R} \left(\frac{1}{2}(\varphi^\delta)'' + (\varphi^\delta)'b + \varphi^\delta \right) u_s \mathrm{d}\lambda\mathrm{d}s - \int_0^t \int_\mathbb{R} \varphi^\delta \mathrm{d}\beta_s\mathrm{d}s$$

Noting $v_0 = u_0 - u_0 = 0$, subtracting the analogous equation for $(\tilde{u}, \tilde{\beta})$ yields

$$\int_{\mathbb{R}} \varphi^\delta v_t \mathrm{d}\lambda = \int_0^t \int_{\mathbb{R}} \left(\frac{1}{2}(\varphi^\delta)'' + (\varphi^\delta)' b + \varphi^\delta \right) v_s \mathrm{d}\lambda \mathrm{d}s - \int_0^t \int_{\mathbb{R}} \varphi^\delta \mathrm{d}\beta_s \mathrm{d}s + \int_0^t \int_{\mathbb{R}} \varphi^\delta \mathrm{d}\tilde{\beta}_s \mathrm{d}s.$$

By antisymmetry of η_δ' and Fubini's theorem, we have

$$\begin{aligned} \int_{\mathbb{R}} (\varphi^\delta)' b v_s \mathrm{d}\lambda &= \int_{\mathbb{R}} \int_{\mathbb{R}} \eta_\delta'(x-y)\varphi(y) b(x) v_s(x) \mathrm{d}y \mathrm{d}x \\ &= -\int_{\mathbb{R}} \int_{\mathbb{R}} \eta_\delta'(y-x)\varphi(y) b(x) v_s(x) \mathrm{d}x \mathrm{d}y = -\int_{\mathbb{R}} \varphi \cdot ((bv_s)^\delta)' \mathrm{d}\lambda. \end{aligned} \tag{11}$$

Together with similar arguments for the remaining terms, the above equation can be rewritten as

$$\int_{\mathbb{R}} \varphi v_t^\delta \mathrm{d}\lambda = \int_0^t \int_{\mathbb{R}} \varphi \left(\frac{1}{2}(v_s^\delta)'' - ((bv_s)^\delta)' + v_s^\delta \right) \mathrm{d}\lambda \mathrm{d}s + \int_0^t \int_{\mathbb{R}} \varphi(\tilde{\beta}_s^\delta - \beta_s^\delta) \mathrm{d}\lambda \mathrm{d}s.$$

Since this holds for every $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$, the fundamental lemma of the calculus of variations yields

$$v_t^\delta = \int_0^t \left(\frac{1}{2}(v_s^\delta)'' - ((bv_s)^\delta)' + v_s^\delta \right) \mathrm{d}s + \int_0^t \tilde{\beta}_s^\delta - \beta_s^\delta \mathrm{d}s.$$

Before continuing, we record a few observations: We have $v \in \mathcal{U}$, which guarantees $\sup_{s\in[0,t]} \|v_s\|_2 < \infty$. Applying Young's convolution inequality and recalling that b is bounded, we deduce

$$\sup_{s\in[0,t]} \|v_s^\delta\|_2 < \infty, \quad \sup_{s\in[0,t]} \|(v_s^\delta)''\|_2 < \infty, \quad \sup_{s\in[0,t]} \|((bv_s)^\delta)'\|_2 < \infty.$$

Moreover, $v \in \mathcal{U}$ guarantees $\sup_{s\in[0,t]} \|v_s^\delta\|_\infty \le \sup_{s\in[0,t]} \|v_s\|_\infty < \infty$. Finally, β_s^δ and $\tilde{\beta}_s^\delta$ are probability densities for all $s \in [0,t]$ since β_s and $\tilde{\beta}_s$ are probability measures. These observations, together with Hölder's inequality, allow us to apply Fubini's theorem in the subsequent calculation. Noting $v_0^\delta = 0$, the above representation of v_t^δ yields

$$\frac{1}{2}\|v_t^\delta\|_2^2 = \int_{\mathbb{R}} \int_0^t v_s^\delta \dot{v}_s^\delta \mathrm{d}s \mathrm{d}\lambda = \int_0^t \int_{\mathbb{R}} v_s^\delta \left(\frac{1}{2}(v_s^\delta)'' - ((bv_s)^\delta)' + v_s^\delta \right) \mathrm{d}\lambda \mathrm{d}s + \int_0^t \int_{\mathbb{R}} v_s^\delta (\tilde{\beta}_s^\delta - \beta_s^\delta) \mathrm{d}\lambda \mathrm{d}s. \tag{12}$$

Our aim now is to prepare an application of Gronwall's lemma. To this end, we consider the terms on the right-hand side of (12) one by one. Since $v \in \mathcal{U}$ implies $v_s \in \mathcal{C}_0(\mathbb{R}, \mathbb{R})$ and hence $v_s^\delta \in \mathcal{C}_0(\mathbb{R}, \mathbb{R})$ for all $s \in [0,t]$, integration by parts yields

$$\int_0^t \int_{\mathbb{R}} v_s^\delta (v_s^\delta)'' \mathrm{d}\lambda \mathrm{d}s = -\int_0^t \int_{\mathbb{R}} (v_s^\delta)'(v_s^\delta)' \mathrm{d}\lambda \mathrm{d}s = -\int_0^t \|(v_s^\delta)'\|_2^2 \mathrm{d}s. \tag{13}$$

Combining a similar integration by parts with the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} -\int_0^t \int_{\mathbb{R}} v_s^\delta ((bv_s)^\delta)' \mathrm{d}\lambda \mathrm{d}s &= \int_0^t \int_{\mathbb{R}} (v_s^\delta)'(bv_s)^\delta \mathrm{d}\lambda \mathrm{d}s \\ &\le \int_0^t \|(v_s^\delta)'\|_2 \|(bv_s)^\delta\|_2 \mathrm{d}s \le \frac{1}{2}\int_0^t \|(v_s^\delta)'\|_2^2 \mathrm{d}s + \frac{1}{2}\int_0^t \|(bv_s)^\delta\|_2^2 \mathrm{d}s. \end{aligned}$$

Further, we get

$$\begin{aligned} \|(bv_s)^\delta\|_2^2 &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \eta_\delta(x-y) b(y) v_s(y) \mathrm{d}y \right|^2 \mathrm{d}x \le \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \eta_\delta(x-y) |b(y)| |v_s(y)| \mathrm{d}y \right)^2 \mathrm{d}x \\ &\le \|b\|_\infty^2 \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \eta_\delta(x-y) |v_s(y)| \mathrm{d}y \right)^2 \mathrm{d}x = \|b\|_\infty^2 \||v_s|^\delta\|_2^2, \qquad s \in [0,t]. \end{aligned}$$

Noting $\|v_s\|_2 = \||v_s|\|_2$, we further obtain

$$\||v_s|^\delta\|_2^2 \le (\||v_s|^\delta - |v_s|\|_2 + \|v_s - v_s^\delta\|_2 + \|v_s^\delta\|_2)^2 \le 2\big(\big\||v_s|^\delta - |v_s|\big\|_2 + \big\|v_s - v_s^\delta\big\|_2\big)^2 + 2\|v_s^\delta\|_2^2, \quad s \in [0,t].$$

Combining these three inequalities, we deduce

$$\begin{aligned} &-\int_0^t \int_{\mathbb{R}} v_s^\delta ((bv_s)^\delta)' \mathrm{d}\lambda \mathrm{d}s \\ &\le \frac{1}{2}\int_0^t \|(v_s^\delta)'\|_2^2 \mathrm{d}s + \|b\|_\infty^2 t \sup_{s\in[0,t]} \big(\big\||v_s|^\delta - |v_s|\big\|_2 + \big\|v_s - v_s^\delta\big\|_2\big)^2 + \|b\|_\infty^2 \int_0^t \|v_s^\delta\|_2^2 \mathrm{d}s. \end{aligned} \tag{14}$$

Let us finally consider the last term in (12). Recalling $\beta \in \mathcal{B}^u$ and $u \geq 0$, we have $\int_{\mathbb{R}} u_s \mathrm{d}\beta_s = \|u_s\|_\infty \geq \int_{\mathbb{R}} u_s \mathrm{d}\tilde{\beta}_s$ for a.e. $s \in [0,t]$. Together with the analogous fact for $\tilde{u}$, we get $\int_{\mathbb{R}} v_s \mathrm{d}(\tilde{\beta}_s - \beta_s) \leq 0$ for a.e. $s \in [0,t]$. Using an argument similar to (11), based on the symmetry of η_δ, in the first step, we thus obtain

$$\begin{aligned}\int_0^t \int_{\mathbb{R}} v_s^\delta(\tilde{\beta}_s^\delta - \beta_s^\delta)\mathrm{d}\lambda\mathrm{d}s &= \int_0^t \int_{\mathbb{R}} (v_s^\delta)^\delta \mathrm{d}(\tilde{\beta}_s - \beta_s)\mathrm{d}s \\ &\leq \int_0^t \int_{\mathbb{R}} v_s \mathrm{d}(\tilde{\beta}_s - \beta_s)\mathrm{d}s + 2\int_0^t \|v_s - (v_s^\delta)^\delta\|_\infty \mathrm{d}s \leq 2t \sup_{s\in[0,t]} \|v_s - (v_s^\delta)^\delta\|_\infty.\end{aligned}$$

Inserting this as well as (13) and (14) into (12), we get

$$\frac{1}{2}\|v_t^\delta\|_2^2 \leq (\|b\|_\infty^2 + 1)\int_0^t \|v_s^\delta\|_2^2 \mathrm{d}s + \|b\|_\infty^2 t \sup_{s\in[0,t]} (\||v_s|^\delta - |v_s|\|_2 + \|v_s - v_s^\delta\|_2)^2 + 2t \sup_{s\in[0,t]} \|v_s - (v_s^\delta)^\delta\|_\infty.$$

Gronwall's lemma yields

$$\|v_t^\delta\|_2^2 \leq \left(2\|b\|_\infty^2 t \sup_{s\in[0,t]} (\||v_s|^\delta - |v_s|\|_2 + \|v_s - v_s^\delta\|_2)^2 + 4t \sup_{s\in[0,t]} \|v_s - (v_s^\delta)^\delta\|_\infty\right) \mathrm{e}^{2(\|b\|_\infty^2+1)t}.$$

Noting $v, |v| \in \mathcal{U}$, Lemma 10 shows that this converges to 0 as $\delta \searrow 0$, proving $\|v_t\|_2 = \lim_{\delta\searrow 0} \|v_t^\delta\|_2 = 0$. Since v_t is continuous, we end up with $u(t,\cdot) = \tilde{u}(t,\cdot)$. □

6 Complementarity Condition and Proof of Theorem 1

One of our main tasks in this final section can roughly be summarized as follows: If (μ, β) is a subsequential distributional limit of $(\mu^{n,\varepsilon}, \beta^{n,\varepsilon})$ as $n \to \infty$ followed by $\varepsilon \searrow 0$ and if u is a continuous density of μ, then the complementarity condition $\beta \in \mathcal{B}^u$ is satisfied a.s. The argument for the limit $\varepsilon \searrow 0$ will be based on uniform convergence of the densities on every finite time horizon.

Lemma 11. *Let $(\mu^m)_{m\in\mathbb{N}} \subseteq \mathcal{C}([0,\infty), \mathcal{M}_1)$ be a sequence which converges to $\mu \in \mathcal{C}([0,\infty), \mathcal{M}_1)$. We assume that μ^m has a density u^m for every $m \in \mathbb{N}$ such that $(u^m|_{[0,T]\times\mathbb{R}})_{m\in\mathbb{N}}$ is uniformly equicontinuous for every $T \geq 0$. For each $T \geq 0$, we additionally assume*[1] $\sup_{m\in\mathbb{N}} \|u^m|_{[0,T]\times\mathbb{R}}\|_\infty < \infty$ *and*

$$\lim_{R\to\infty} \sup_{m\in\mathbb{N}} \sup_{t\in[0,T]} \sup_{x\in[-R,R]^c} |u^m(t,x)| = 0. \tag{15}$$

Then μ has a density $u \in \mathcal{C}([0,\infty)\times\mathbb{R}, \mathbb{R})$ satisfying

$$\lim_{m\to\infty} \|(u^m - u)|_{[0,T]\times\mathbb{R}}\|_\infty = 0 \quad \textit{and} \quad \lim_{R\to\infty} \sup_{t\in[0,T]} \sup_{x\in[-R,R]^c} |u(t,x)| = 0, \quad T \geq 0. \tag{16}$$

Moreover, if we also have

$$\lim_{R\to\infty} \sup_{m\in\mathbb{N}} \sup_{t\in[0,T]} \int_{[-R,R]^c} |u^m(t,x)|^2 \mathrm{d}x = 0, \tag{17}$$

then $u \in \mathcal{U}$ holds.

Proof. The Arzelà-Ascoli theorem (combined with exhausting $[0,\infty)\times\mathbb{R}$ by a sequence of compact sets and a diagonalization argument) implies that every subsequence of $(u^m)_{m\in\mathbb{N}}$ has a sub-subsequence converging locally uniformly to some $u \in \mathcal{C}([0,\infty)\times\mathbb{R}, \mathbb{R})$. Now let $t \geq 0$. On the one hand, convergence of $(\mu^m)_{m\in\mathbb{N}}$ to μ yields

$$\int_{\mathbb{R}} \varphi(x) u^m(t,x)\mathrm{d}x = \int_{\mathbb{R}} \varphi(x)\mathrm{d}\mu_t^m(x) \xrightarrow{m\to\infty} \int_{\mathbb{R}} \varphi(x)\mathrm{d}\mu_t(x), \quad \varphi \in \mathcal{C}_c(\mathbb{R},\mathbb{R}).$$

On the other hand, the left-hand side converges to $\int_{\mathbb{R}} \varphi(x)u(t,x)\mathrm{d}x$ for every $\varphi \in \mathcal{C}_c(\mathbb{R},\mathbb{R})$ along the sub-subsequence from above. Consequently, u is a density of μ. Since μ can have at most one continuous density, the whole sequence $(u^m)_{m\in\mathbb{N}}$ must converge locally uniformly to u by a standard sub-subsequence argument. Together with (15), we obtain (16). The final claim is an immediate consequence of the convergence just established, Fatou's lemma and the definition of $\mathcal{U}$. □

We proceed with the announced auxiliary result concerning the complementarity condition.

[1] These additional assumptions are actually implied by the properties stated before. However, since we already know that the additional assumptions are satisfied in our setting, we simply include them to shorten the proof.

Lemma 12. *For every $m \in \mathbb{N}$, let (μ^m, β^m) be any subsequential distributional limit of $(\mu^{n,1/m}, \beta^{n,1/m})_{n\in\mathbb{N}}$. Further, let (μ, β) be any subsequential distributional limit of $(\mu^m, \beta^m)_{m\in\mathbb{N}}$ and let u a.s. be a (random) continuous density of μ. Then $\beta \in \mathcal{B}^u$ holds a.s.*

Proof. Throughout the proof, let $T > 0$ be fixed. For all $n, m \in \mathbb{N}$, we introduce

$$v^{n,m} : [0,T] \times \mathbb{R} \to [0,1], \quad (t,x) \mapsto \begin{cases} \mu^{n,1/m}_{t-}(C^{1/m}_k), & x \in (\frac{k}{m}, \frac{k+1}{m}) \text{ for some } k \in \mathbb{Z}, \\ \max\{\mu^{n,1/m}_{t-}(C^{1/m}_{k-1}), \mu^{n,1/m}_{t-}(C^{1/m}_k)\}, & x = \frac{k}{m} \quad \text{for some } k \in \mathbb{Z}, \end{cases}$$

with the convention $\mu^{n,1/m}_{0-} := \mu^{n,1/m}_0$. Noting

$$\mathbb{Z} \setminus A^{1/m}(X^{n,1/m}(t-)) = \left\{ k \in \mathbb{Z} : \mu^{n,1/m}_{t-}(C^{1/m}_k) < \max_{k'\in\mathbb{Z}} \mu^{n,1/m}_{t-}(C^{1/m}_{k'}) \right\}, \quad t > 0,\ n, m \in \mathbb{N},$$

we, by construction, have

$$\begin{aligned} &\beta^{n,1/m}(\{(t,x) \in [0,T] \times \mathbb{R} : v^{n,m}(t,x) < \|v^{n,m}(t,\cdot)\|_\infty\}) \\ &\le \beta^{n,1/m}(\{0\} \times \mathbb{R}) + \beta^{n,1/m}\left(\left\{ (t,x) \in (0,T] \times \mathbb{R} : x \in \bigcup_{k\in\mathbb{Z}\setminus A^{1/m}(X^{n,1/m}(t-))} C^{1/m}_k \right\} \right) = 0, \quad n, m \in \mathbb{N}, \end{aligned}$$

and thus

$$\int_{[0,T]\times\mathbb{R}} (\|v^{n,m}(t,\cdot)\|_\infty - v^{n,m}(t,x)) \mathrm{d}\beta^{n,1/m}(t,x) = 0, \quad n, m \in \mathbb{N}. \tag{18}$$

Now fix $m \in \mathbb{N}$. Our intermediate goal is to prove a similar concentration of β^m. Let $(n^m_l)_{l\in\mathbb{N}} \subseteq \mathbb{N}$ be a sequence of indices such that (μ^m, β^m) is the distributional limit of $(\mu^{n^m_l,1/m}, \beta^{n^m_l,1/m})_{l\in\mathbb{N}}$. For the rest of this paragraph, we assume sure convergence by invoking Skorokhod's representation theorem and discarding the exceptional set. All statements are to be understood pointwise on the new sample space. As $l \to \infty$, we claim that $v^{n^m_l,m}$ converges uniformly to

$$v^m : [0,T] \times \mathbb{R} \to [0,1], \quad (t,x) \mapsto \begin{cases} \mu^m_t(C^{1/m}_k), & x \in (\frac{k}{m}, \frac{k+1}{m}) \text{ for some } k \in \mathbb{Z}, \\ \max\{\mu^m_t(C^{1/m}_{k-1}), \mu^m_t(C^{1/m}_k)\}, & x = \frac{k}{m} \quad \text{for some } k \in \mathbb{Z}. \end{cases}$$

To prove this, let d_{LP} be the Lévy-Prohorov metric on $\mathcal{M}_1$. Since Proposition 5 guarantees $\mu^m \in \mathcal{C}([0,\infty), \mathcal{M}_1)$, a classic property of Skorokhod spaces yields $\lim_{l\to\infty} \sup_{t\in[0,T]} d_{\mathrm{LP}}(\mu^{n^m_l,1/m}_t, \mu^m_t) = 0$. Noting that $\mu^m|_{[0,T]}$ is uniformly continuous, this implies that $(\eta_l)_{l\in\mathbb{N}} := (\sup_{t\in[0,T]} d_{\mathrm{LP}}(\mu^{n^m_l,1/m}_{t-}, \mu^m_t))_{l\in\mathbb{N}}$ is a null sequence, essentially by means of the triangle inequality. Since μ^m has a density which is bounded on $[0,T] \times \mathbb{R}$ by Lemma 6, we have $\lim_{\eta\searrow 0} \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \mu^m_t((x, x+3\eta)) = 0$. Now let $l \in \mathbb{N}$. By definition of d_{LP} and C^m_k, we get

$$\begin{aligned} \mu^m_t(C^m_k) &\le \mu^{n^m_l,1/m}_{t-}(C^m_k) + \mu^{n^m_l,1/m}_{t-}((\tfrac{k}{m} - \eta_l, \tfrac{k}{m})) + \mu^{n^m_l,1/m}_{t-}([\tfrac{k+1}{m}, \tfrac{k+1}{m} + \eta_l)) + \eta_l \\ &\le \mu^{n^m_l,1/m}_{t-}(C^m_k) + \mu^m_t((\tfrac{k}{m} - 2\eta_l, \tfrac{k}{m} + \eta_l)) + \mu^m_t((\tfrac{k+1}{m} - \eta_l, \tfrac{k+1}{m} + 2\eta_l)) + 3\eta_l, \qquad t \in [0,T],\ k \in \mathbb{Z}, \end{aligned}$$

and

$$\mu^{n^m_l,1/m}_{t-}(C^m_k) \le \mu^m_t(C^m_k) + \mu^m_t((\tfrac{k}{m} - \eta_l, \tfrac{k}{m})) + \mu^m_t([\tfrac{k+1}{m}, \tfrac{k+1}{m} + \eta_l)) + \eta_l, \quad t \in [0,T],\ k \in \mathbb{Z}.$$

Recalling $\lim_{l\to\infty} \eta_l = 0$, we thus obtain

$$\limsup_{l\to\infty} \sup_{t\in[0,T]} \sup_{k\in\mathbb{Z}} \left| \mu^{n^m_l,1/m}_{t-}(C^m_k) - \mu^m_t(C^m_k) \right| \le \lim_{l\to\infty} \left(2 \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \mu^m_t((x, x+3\eta_l)) + 3\eta_l \right) = 0.$$

This implies that $v^{n^m_l,m}$ converges uniformly to v^m as $l \to \infty$, as claimed. Together with (18) and $\beta^{n^m_l,1/m} \in \mathcal{M}_{\nearrow 1}$ for all $l \in \mathbb{N}$, we deduce

$$\begin{aligned} &\lim_{l\to\infty} \int_{[0,T]\times\mathbb{R}} (\|v^m(t,\cdot)\|_\infty - v^m(t,x)) \mathrm{d}\beta^{n^m_l,1/m}(t,x) \\ &= \lim_{l\to\infty} \int_{[0,T]\times\mathbb{R}} \big((\|v^m(t,\cdot)\|_\infty - v^m(t,x)) - (\|v^{n^m_l,m}(t,\cdot)\|_\infty - v^{n^m_l,m}(t,x)) \big) \mathrm{d}\beta^{n^m_l,1/m}(t,x) = 0. \end{aligned}$$

Finally, since the integrand is lower semicontinuous and non-negative, the Portmanteau theorem yields

$$\int_{[0,T]\times\mathbb{R}} (\|v^m(t,\cdot)\|_\infty - v^m(t,x))\mathrm{d}\beta^m(t,x) = 0 \quad \text{a.s.} \tag{19}$$

Let us now turn toward the limit as $m \to \infty$. As above, we invoke Skorokhod's representation theorem to assume that $(\mu^m, \beta^m)_{m\in\mathbb{N}}$ surely converges to (μ, β) along a subsequence. Lemmas 6, 7 and 8 imply that μ^m has a (random) continuous density u^m for each $m \in \mathbb{N}$ such that $(u^m|_{[0,T]\times\mathbb{R}})_{m\in\mathbb{N}}$ is uniformly equicontinuous and uniformly bounded with (15). Thus Lemma 11 shows that μ a.s. has a (random) density $\tilde{u} \in \mathcal{C}([0,\infty)\times\mathbb{R}, \mathbb{R})$ such that $(u^m|_{[0,T]\times\mathbb{R}})_{m\in\mathbb{N}}$ converges uniformly to $\tilde{u}|_{[0,T]\times\mathbb{R}}$ along the subsequence along which $(\mu^m, \beta^m)_{m\in\mathbb{N}}$ converges to (μ, β). Since μ can have at most one continuous density, we must a.s. have $\tilde{u} = u$. Now let $w : (0,\infty) \to [0,\infty)$ be a joint modulus of continuity for $(u^m|_{[0,T]\times\mathbb{R}})_{m\in\mathbb{N}}$. By construction of v^m and by the mean value theorem for integrals, we a.s. obtain $\|u^m|_{[0,T]\times\mathbb{R}} - mv^m\|_\infty \le w(\frac{1}{m})$ for all $m \in \mathbb{N}$, showing that $(u^m|_{[0,T]\times\mathbb{R}} - mv^m)_{m\in\mathbb{N}}$ a.s. converges to 0. Together with (19) and $\beta^m \in \mathcal{M}_{\nearrow 1}$ for all $m \in \mathbb{N}$, we deduce that

$$\begin{aligned}
&\int_{[0,T]\times\mathbb{R}} (\|u(t,\cdot)\|_\infty - u(t,x))\mathrm{d}\beta^m(t,x)\\
\le &\int_{[0,T]\times\mathbb{R}} \big((\|u(t,\cdot)\|_\infty - u(t,x)) - (\|u^m(t,\cdot)\|_\infty - u^m(t,x))\big)\mathrm{d}\beta^m(t,x)\\
&+ \int_{[0,T]\times\mathbb{R}} ((\|u^m(t,\cdot)\|_\infty - u^m(t,x)) - m(\|v^m(t,\cdot)\|_\infty - v^m(t,x)))\,\mathrm{d}\beta^m(t,x)
\end{aligned}$$

a.s. converges to 0 as $m \to \infty$ along the subsequence from above. Once again, the Portmanteau theorem yields

$$\int_{[0,T]\times\mathbb{R}} (\|u(t,\cdot)\|_\infty - u(t,x))\mathrm{d}\beta(t,x) = 0 \quad \text{a.s.}$$

Since $T > 0$ was arbitrary, we must have

$$\beta(\{(t,x) \in [0,\infty)\times\mathbb{R} : u(t,x) < \|u(t,\cdot)\|_\infty\}) = 0 \quad \text{a.s.},$$

as claimed. □

We continue with a non-standard version of a classic diagonalization result for limits of double sequences, which will be the top-level ingredient of our proof of Theorem 1.

Lemma 13. *Let $(\mathcal{E}, d)$ be a compact metric space and let $(z_{n,m})_{n,m\in\mathbb{N}} \subseteq \mathcal{E}$. Assume that there is some $z \in \mathcal{E}$ with the following property: If $z_m \in \mathcal{E}$ is any accumulation point of $(z_{n,m})_{n\in\mathbb{N}}$ for every $m \in \mathbb{N}$, the sequence $(z_m)_{m\in\mathbb{N}}$ converges to z. Then there exists a sequence $(m_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ with $\lim_{n\to\infty} m_n = \infty$ such that $(z_{n,m_n})_{n\in\mathbb{N}}$ converges to z.*

Proof. Let $l \in \mathbb{N}$. By assumption, there must be an $m'_l \in \mathbb{N}$ such that, for every $m \in \mathbb{N}$ with $m \ge m'_l$, every accumulation point $z_m \in \mathcal{E}$ of $(z_{n,m})_{n\in\mathbb{N}}$ satisfies $d(z_m, z) < \frac{1}{2l}$. Due to compactness of $\mathcal{E}$, this implies that, for every $m \in \mathbb{N}$ with $m \ge m'_l$, there must be an $n_{l,m} \in \mathbb{N}$ with $d(z_{n,m}, z) < \frac{1}{l}$ for all $n \in \mathbb{N}$ with $n \ge n_{l,m}$. Using the conventions $\max\emptyset := 0$ and $[0] := \emptyset$, we set

$$n_m := m + \max\{n_{l,j} : l, j \in [m],\, j \ge m'_l\}, \quad m \in \mathbb{N},$$

and define

$$m_n := m, \quad n \in [n_m] \setminus [n_{m-1}],\, m \in \mathbb{N}.$$

Since $(n_m)_{m\in\mathbb{N}}$ is strictly increasing, $(m_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ is a well-defined non-decreasing sequence with $\lim_{n\to\infty} m_n = \infty$. Now let $l \in \mathbb{N}$ and $\bar{m} := \max\{l, m'_l\}$. Further, let $n \in \mathbb{N}$ with $n \ge n_{\bar{m}}$. Then we have $m_n \ge \bar{m} \ge m'_l$ and thus $n \ge n_{\bar{m}} \ge n_{l,m_n}$, proving $d(z_{n,m_n}, z) < \frac{1}{l}$. □

Finally, we are ready to put everything together.

Proof of Theorem 1. For a Polish space E, let $\mathcal{M}_1(E)$ be the Polish space of all Borel probability measures on E. Further, for a random variable Z, let $\mathcal{L}(Z)$ be its law. As a consequence of Proposition 5 and Prohorov's theorem, the metrizable space

$$\mathcal{E} := \overline{\{\mathcal{L}(\mu^{n,\varepsilon}, \beta^{n,\varepsilon}) : n \in \mathbb{N},\, \varepsilon > 0\}} \subseteq \mathcal{M}_1(\mathcal{D}([0,\infty), \mathcal{M}_1) \times \mathcal{M}_{\nearrow 1})$$

is compact. Now let $m \in \mathbb{N}$ and let (μ^m, β^m) be any subsequential distributional limit of $(\mu^{n,1/m}, \beta^{n,1/m})_{n\in\mathbb{N}}$. (Note that this is equivalent to $(\mu^m, \beta^m) \sim \mathbb{Q}_m$ for some accumulation point $\mathbb{Q}_m \in \mathcal{E}$ of $(\mathcal{L}(\mu^{n,1/m}, \beta^{n,1/m}))_{n\in\mathbb{N}}$.)

Moreover, let $(\tilde{\mu}, \tilde{\beta})$ be a subsequential distributional limit of $(\mu^m, \beta^m)_{m\in\mathbb{N}}$. Invoking Skorokhod's representation theorem, we assume a.s. convergence. Lemmas 6, 7 and 8 imply that μ^m has a (random) continuous density u^m with

$$\begin{aligned}&\int_{\mathbb{R}} \varphi(x)u^m(t,x)\mathrm{d}x\\ &= \int_{\mathbb{R}} \varphi(x)u_0(x)\mathrm{d}x - \int_{[0,t]\times\mathbb{R}} \varphi(x)\mathrm{d}\beta^m(s,x)\\ &\quad + \int_0^t \int_{\mathbb{R}} \left(\frac{1}{2}\varphi''(x) + \varphi'(x)b(x) + \varphi(x)\right) u^m(s,x)\mathrm{d}x\mathrm{d}s, \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{R}),\, t\geq 0, \quad \text{a.s.,} \end{aligned} \tag{20}$$

and that $(u^m|_{[0,T]\times\mathbb{R}})_{m\in\mathbb{N}}$ is a.s. uniformly equicontinuous and uniformly bounded with (15) and (17) for all $T \geq 0$. Since Proposition 5 a.s. yields $\mu^m \in \mathcal{C}([0,\infty), \mathcal{M}_1)$, we a.s. have $\tilde{\mu} \in \mathcal{C}([0,\infty), \mathcal{M}_1)$ as well. Lemma 11 a.s. shows that $\tilde{\mu}$ has a (random) density $\tilde{u} \in \mathcal{U}$ with (16). Taking the subsequential limit $m \to \infty$ in (20), using the dominated convergence theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}} \varphi(x)\tilde{u}(t,x)\mathrm{d}x = &\int_{\mathbb{R}} \varphi(x)u_0(x)\mathrm{d}x - \int_{[0,t]\times\mathbb{R}} \varphi(x)\mathrm{d}\tilde{\beta}(s,x)\\ &+ \int_0^t \int_{\mathbb{R}} \left(\frac{1}{2}\varphi''(x) + \varphi'(x)b(x) + \varphi(x)\right) \tilde{u}(s,x)\mathrm{d}x\mathrm{d}s, \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{R}),\, t\geq 0, \quad \text{a.s.}\end{aligned}$$

Together with Lemma 12, this shows that $(\tilde{u}, \tilde{\beta}) \in \mathcal{U} \times \mathcal{M}_{\nearrow 1}$ solves (3) a.s. In particular, a solution exists, and Proposition 9 ensures that it is unique. Denoting this solution by (u, β), uniqueness gives $(\tilde{u}, \tilde{\beta}) = (u, \beta)$ a.s. Letting $\mu \in \mathcal{C}([0,\infty), \mathcal{M}_1)$ with density u, this also implies $\tilde{\mu} = \mu$ a.s. Recalling that convergence in distribution to a deterministic limit implies convergence in probability, Lemma 13 finally yields the claim. □

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References

[AMW26] Rami Atar, Leonid Mytnik, and Gershon Wolansky. Trimmed branching random walk and a free obstacle problem, 2026. arXiv:2601.21054.

[Ata25] Rami Atar. A weak formulation of free boundary problems and its application to hydrodynamic limits of particle systems with selection. *The Annals of Probability*, 53(5):1825–1856, 2025.

[BBNP22] Julien Berestycki, Eric Brunet, James Nolen, and Sarah Penington. Brownian bees in the infinite swarm limit. *The Annals of Probability*, 50(6):2133–2177, 2022.

[BBP19] Julien Berestycki, Eric Brunet, and Sarah Penington. Global existence for a free boundary problem of Fisher-KPP type. *Nonlinearity*, 32(10):3912–3939, 2019.

[BDMM06] Eric Brunet, Bernard Derrida, Alfred H. Mueller, and Stephane Munier. Noisy traveling waves: effect of selection on genealogies. *Europhysics Letters*, 76(1):1–7, 2006.

[BDMM07] Éric Brunet, Bernard Derrida, Alfred H. Mueller, and Stéphane Munier. Effect of selection on ancestry: An exactly soluble case and its phenomenological generalization. *Physical Review E*, 76(4):041104, 2007.

[BPT25] Julien Berestycki, Sarah Penington, and Oliver Tough. Convergence and front position for an FKPP-type free boundary problem, 2025. arXiv:2511.23457.

[BT24] Julien Berestycki and Oliver Tough. Selection principle for the n-BBM, 2024. To appear in The Annals of Probability; arXiv:2407.05792.

[DM82] Claude Dellacherie and Paul-André Meyer. *Probabilities and potential. B*, volume 72 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1982. Theory of martingales, Translated from the French by J. P. Wilson.

[DFPS19] Anna De Masi, Pablo A. Ferrari, Errico Presutti, and Nahuel Soprano-Loto. Hydrodynamics of the n-BBM process. In *Stochastic Dynamics Out of Equilibrium*, volume 282 of *Springer Proceedings in Mathematics & Statistics*, pages 523–549. Springer, Cham, 2019.

[DR11] Rick Durrett and Daniel Remenik. Brunet-Derrida particle systems, free boundary problems and Wiener-Hopf equations. *The Annals of Probability*, 39(6):2043–2078, 2011.

[EK86] Stewart N. Ethier and Thomas G. Kurtz. *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1986. Characterization and convergence.

[FM04] Nicolas Fournier and Sylvie Meleard. A microscopic probabilistic description of a locally regulated population and macroscopic approximations. *The Annals of Applied Probability*, 14(4):1880–1919, 2004.

[HL04] Emilio Hernández-García and Cristóbal López. Clustering, advection, and patterns in a model of population dynamics with neighborhood-dependent rates. *Physical Review E*, 70(1):016216, 2004.

[HR17] Simon C. Harris and Matthew I. Roberts. The many-to-few lemma and multiple spines. *Ann. Inst. Henri Poincaré Probab. Stat.*, 53(1):226–242, 2017.

[JS03] Jean Jacod and Albert N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2003.

[Kal17] Olav Kallenberg. *Random measures, theory and applications*, volume 77 of *Probability Theory and Stochastic Modelling*. Springer, Cham, 2017.

[Mai16] Pascal Maillard. Speed and fluctuations of n-particle branching Brownian motion with spatial selection. *Probability Theory and Related Fields*, 166:1061–1173, 2016.

[QZ02] Zhongmin Qian and Weian Zheng. Sharp bounds for transition probability densities of a class of diffusions. *Comptes Rendus. Mathématique*, 335(11):953–957, 2002.

[Ver80] A. Ju. Veretennikov. Strong solutions and explicit formulas for solutions of stochastic integral equations. *Mat. Sb. (N.S.)*, 111(153)(3):434–452, 480, 1980.